CERME 6 – WORKING GROUP 9

DIFFERENT THEORETICAL PERSPECTIVES AND APPROACHES IN MATHEMATICS EDUCATION RESEARCH

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INTRODUCTION

DIFFERENT THEORETICAL PERSPECTIVES AND APPROACHES IN MATHEMATICS EDUCATION RESEARCH - STRATEGIES AND DIFFICULTIES WHEN CONNECTING THEORIES

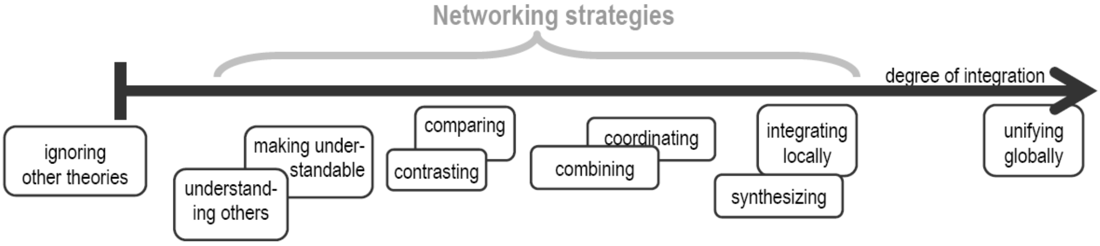
Susanne Prediger[1], Marianna Bosch[2], Ivy Kidron[3], John Monaghan[4], Gérard Sensevy[5]

[1]Dortmund University (Germany), [2] Universitat Ramon Llull (Spain), [3] Jerusalem College of Technology (Israel),
[4]University of Leeds (United Kingdom), [5] INRP Lyon (France)

A large diversity of different theoretical perspectives and research paradigms characterize the European mathematics education research community. Since CERME 4, the ‘Theory Working Group’ has explored differences between these theories, their expression in different research practices and possible ways to deal with this diversity (see Artigue et al. 2006, Bosch et al. 2008 and Prediger et al. 2008).

Exploiting diversity as a rich resource for grasping complex realities (Bikner-Ahsbahs & Prediger 2006) requires developing strategies for connecting theories or research results obtained using different theoretical approaches. In 2007, the Theory Working Group continued its efforts in this direction and reflected on opportunities and difficulties of what we call ‘networking theories’. We noted different intentions behind researchers efforts to network theories. In some cases, the goal is to investigate the complementary insights that are offered when we analyze given data with different theories (Kidron, 2008). In other cases, the intention is to explore the insights offered by each theory to the other theories and, at the same time, to highlight the limits of such an endeavour (Kidron et al., 2008; Radford, 2008).

The call for papers for the Theory Working Group at CERME 6 was guided by the idea of avoiding an overly abstract discussion without a concrete basis. That is why we called for papers with *concrete case studies* in which two or more theoretical approaches were connected. After an intensive peer review process, 15 substantial papers were chosen for discussion in the working group and for publication in these proceedings. The most important issues arising in the discussion of these case studies can be sketched under some key words structured according to the landscape of networking strategies as proposed by Prediger, Bikner-Ahsbahs & Arzarello, 2008).



Main issues arising in comparing and contrasting: Dimension of comparison

Comparing theories requires categories for comparison. A variety of categories have been suggested by Prediger, Arzarello & Bikner-Ahsbahs (2008). The discussion this year was influence by the following:

- the delimitation of empirical data and the kind of questions that arise, as well as the concrete formulation of results (see Ligozat & Schubauer-Leoni in this volume);
- the distinction between theoretical approaches and perspectives (discussed by Wedege in this volume);
- an ontological characterization of theories such as that proposed by Winsløw (in this volume) called the *GOA*-Model, which distinguishes theories according to nature of their objects of research, namely groups (*G*) structured by certain relationships, the organisation (*O*) of knowledge and practice, and artefacts (*A*) used to access and communicate in and about *O*.
- an epistemological characterization of theories such as that proposed by Radford (2008), distinguishing between their basic principles, their methodology, and the paradigmatic questions that are approached.

Main issues arising in combining and coordinating: Compatibility

In order to combine or coordinate different theories, it appears to us that the theories must, in some sense, be compatible; but what exactly does this mean? In working group discussions of the case studies presented in the papers, different levels were posited as possible locations for potential incompatibilities:

- the level of general principles, e.g. epistemological principles about how to interpret mathematical knowledge;
- the level of basic 'paradigms', the potential danger of hastily combining stability-oriented with transformation–oriented perspectives;
- the level of central constructs: although the sense or denotation of constructs may not be identical over different theories, they should not be contradictory (Gellert in this volume shows an interesting example of networking around the construct "rules");
- the level of practical consequences: if coordinating theories in empirical work leads to contradictory practical consequences with regard to learning, then there is a need to continue reflection (see Bergsten & Jablonka in this volume);
- the level of ontology: this does not seem to present as many difficulties as some of the above since different grain sizes of analyses and focuses might help in combining theories (see, for example, Jungwirth in this volume).

In the working group discussion it was suggested that when paradigmatic research questions and/or objects diverge in different perspectives, the combination of these perspectives in the course of analysing an empirical phenomenon might produce incommensurable, but not contradictory, results, as shown by the paper of Bergsten & Jablonka (in this volume). This raises the question of whether it is acceptable that different results can, without contradiction, lead to radically opposed interpretations.

On the other hand, we found some aspects that facilitate the connection of theories. Theories might be linked more easily when they are not too strong with respect to their grammar or their methodologies (i.e. when they are at an early level of elaboration) or when they are complementary with respect to their hypothetical scope or empirical load (see Jungwirth in this volume).

Main issue arising in integrating and synthesizing: Substrategies

The working group discussion regarding strategies for integrating and synthesizing theories led to the tentative proposal to identify substrategies which included: 'bricolaging' (that is adapting non-conflicting principles, notions or local analysis methods of different grand theories); 'subordinating' (see Gellert); 'zooming in and out' (see Jungwirth); and 'metaphorical structuring', the use of single concepts based on metaphors from one theory that converge into another (see Gellert with regard to rules).

As Radford (2008) stated, although connections between theories are possible, there is a *limit* to what can be connected and this limit is determined by the *goal* of the connection and the *specificities* of the theories that are being connected. In the following, we differentiate between different goals in the networking process.

Networking with different aims

In order to link theories beyond comparing and contrasting, we discussed the aims of the papers.:

- Some of the papers propose networking strategies with the aim of understanding an empirical phenomenon that seems difficult to entirely grasp within one single theory. These can be described as having an initial combining strategy that ends up with the construction of local coherence between the notions or principles used. In this sense, Arzarello, Bikner and Sabena (in this volume) combine theories for analysing data about a failed teaching strategy and integrating them (very) locally for the purpose of making sense of the situation described. The paper of Schäfer (in this volume) combines theories for constructing a local theory that improved his potential to approach a 'practical' question about low achieving students. Wedege (in this volume) presents a study in which some aspects of two theoretical perspectives are coordinated. Stadler (in this volume) coordinates different perspectives within one empirical study, describing how a research interest in the transition between mathematics studies at secondary and tertiary levels generates the need for different theoretical approaches.
- A different goal presented by some papers is to network with the aim of dealing with new problems. For example Ligozat & Schubauer-Leoni's and Sensevy's papers are hybrids which borrow constructs from distinct theories for local integra-

tion with conversions in order to address specific research problem, the issue of joint action of the teacher and the students.

- Networking is also an important tool to elaborate existing theories with the aim of increasing their scope by questioning them from the outside. Artigue, Bosch, Gascón & Lenfant (in this volume) show how a theory can evolve locally when an effort is made to approach a question formulated by another theory. The strategy here is to work within one theoretical framework and develop it in interaction with others, for instance by enlarging the set of paradigmatic research questions or its empirical unit of analysis. The work of Jungwirth (in this volume) presents a method of synthesizing local theories for 'zooming in and out' of the data.

- Other papers consider networking with the aim of satisfying the need for an enlarged framework in relation to some new domain of research, assuming the existing frames are insufficient. For instance, Lagrange & Monaghan (in this volume) incorporated Saxe's four parameters model in order to understand the situation of teachers using technology. To these authors, the existing frameworks they considered for viewing teachers' activities in technology-based lessons are insufficient because they focus on teachers' established routines but technology interferes with these routines.

Different kinds of dialogues

Within these aims we may distinguish different kinds of *dialogues* between theories. We use the word 'dialogue' not only to describe that which enables mutual understanding in the way we communicate our theories but also to emphasize differences in the use of language. Different kinds of dialogues were offered in the papers by Ligozat & Schubauer-Leoni, by Sensevy and by Artigue et al. One important characterization is that the dialogues in these papers are between neighbouring approaches - theoretical approaches which were born in the same educational and didactic culture, which may be considered as belonging to the same 'paradigm'. Even so, when we explore the dialogues in depth important differences between the theories can be seen and some interesting questions arise:

- Do these "neighbouring approaches" use the same words with the same meanings? For instance, is the word *milieu* in the Anthropological Theory of the Didactic (ATD) equivalent to the *a didactic milieu* in the Theory of Didactic Situations (TDS)? The same question could be asked in relation to other terms, e.g. *institution* or *contract*. The question could arise also for theories which are not necessarily *neighbouring approaches*.

- Do the different theories deal with different ways of addressing similar issues? For instance, comparing the Joint Action Theory in Didactics (JATD), as described in both Ligozat & Schubauer-Leoni and Sensevy's papers, with ATD and TDS, we may ask what is the difference between *ATD media milieu dialectic*, *TDS a didactic and didactic situations*, and *JATD dialectic between contract and milieu*. Sensevy states that in order to situate JATD in relation to TDS and ATD it can be argued that whereas these two theories initially focus, from a logical point of view, on the nature of *knowledge* (what is the knowledge which is taught?), JATD initially focuses on the *diffusion process* (what is going on when a specific piece of knowledge is taught?). The aim of the networking is to construct a new theory JATD which makes use of existing theories, ATD and TDS. Therefore we may ask what supplementary insights and/or what new questions/problems are offered to ATD and TDS by JATD's analysis of the *diffusion process*? For example, the JATD may raise the following question: within the contract-milieu dialectic how may the teacher link the topogenesis and the chronogenesis processes with respect to the piece of knowledge at stake, and how might these processes lead the teacher, in specific cases, to enact a new learning game? In this question there are some notions from ATD and TDS which are reconceptualized in that they are used in a new way, and there is a new notion (learning game). From an abstract viewpoint, this kind of question is not impossible in ATD and TDS, and it is clearly understandable in these two theories. But the probability that this question is raised in these two theories is not high because their fundamental concerns are not focused on the problems of didactic joint action even though they are interested in didactic action.

In Artigue et al. (in this volume) the notion of 'minimal unit of analysis' appears as a basic aspect of the modelling of educational phenomena proposed by each theory. Starting from the way each perspective reformulates a given research question, we could specify what units of analysis are considered in each case and how they can be connected. The authors add that this could be a good way to improve our capacity for describing and comparing not only the concrete research or practical problem formulated by each theory but also the types of problems that can be proposed, the kind of empirical data needed and the set of 'acceptable answers' that can be provided. When we choose a *specific unit of analysis*, we make decisions not only about the empirical data we consider but also about our different priorities with regard to the focus of the analysis (Bosch & Gascón, 2005).

Final remarks

The discussions that took place in our working group about affordances and constraints of different networking strategies made us aware that the theoretical frameworks used in our research are 'living entities' that evolve through our studies. Some have been around and have developed for many decades, others are less mature. They are our working tools, providing us with new ways of looking at reality, new descriptions of empirical phenomena, new methods of analysis and new possible answers to the difficulties of teaching and learning mathematics. They are imbedded in researchers' social, cultural and institutional inheritances and their development is also impregnated with the personal interactions between researchers and the cooperative work done in our community. When we embody 'theories' into research practices that, at the same time, use theories and produce them, it becomes clear that our reflections about ‘networking theories’ are methodological reflections, referring to the kind of tools we can or cannot use, the basis and the aim of our research, as well as the kind of rules we follow.

Considering the networking of theories as the *networking of research practices* may lead us further not only in our capacity to collaborate between different groups of researchers (and thus accumulate efforts and results) but also to gain insight about the very nature – and the rationale – of our own research in mathematics education.

References

Artigue, M.; Bartolini-Bussi, M., Dreyfus, T., Gray, E., and Prediger, S.: 2006, ‘Different theoretical perspectives and approaches in research in mathematics education’, in M. Bosch. (ed.), *Proceedings of CERME 4*, Sant Feliu de Guíxols 2005, Fundemi IQS – Universitat 2006, 1239-1244.

Arzarello, F.; Bosch, M.; Lenfant, A.; and Prediger, S.: 2008, ‘Different theoretical perspectives in research from teaching problems to research problems’, in D. Pitta-Pantazi and G. Phillipou (eds.), *Proceedings of CERME 5*, Cyprus 2007, ERME, 1618-1627. (+ other contributions of the working group in the same proceedings).

Bikner-Ahsbahs, A. and Prediger, S.: 2006, ‘Diversity of Theories in Mathematics Education - How can we deal with it?’ *ZDM - The International Journal on Mathematics Education* 38(1), 52-57.

Bosch, M., Gascón, J. (2005). La praxéologie comme unité d’analyse des processus didactiques. *In*: Mercier, A., Margolinas, C. (coord.), *Balises en Didactique des Mathématiques*, Grenoble: La Pensée Sauvage (pp. 107-122).

Kidron, I. (2008). Abstraction and consolidation of the limit procept by means of instrumented schemes: the complementary role of three different frameworks. *Educational Studies in Mathematics* 69(3), 197-216. Springer.

Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M. and Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *Zentralblatt fur Didaktik der Mathematik*, 40(2), 247-264. Springer.

Prediger, S,; Bikner-Ahsbahs, A.¸ Arzarello, F.: 2008, ‘Networking strategies and methods for connecting theoretical approaches – First steps towards a conceptual framework’, *ZDM - The International Journal on Mathematics Education*, 40 (2), 165-178.

Prediger, S.; Arzarello, F.; Bosch, M.; Lenfant, A.: 2008, eds., ‘Comparing, Combining, Coordinating - Networking strategies for connecting theoretical approaches’, *ZDM - The International Journal on Mathematics Education*, 40 (2).

Radford, L.:2008, 'Connecting theories in mathematics education: challenges and possibilities', *ZDM - The International Journal on Mathematics Education*, 40 (2).

RESEARCH PROBLEMS EMERGING FROM A TEACHING EPISODE: A DIALOGUE BETWEEN TDS AND ATD

M. Artigue,[I] M. Bosch,[II] J. Gascón [III] & A. Lenfant [IV]

[I]Université Paris 7 (France), [II]Universitat Ramon Llull (Spain),

[III]Universitat Autònoma de Barcelona (Spain), [IV]IUFM de Reims (France)

When approaching an empirical teaching episode or data related to it, theoretical approaches always select and highlight some aspects in detriment of others, globally interpreting the episode using their own conceptual categories and methodological tools. Therefore, different theoretical approaches often construct different research problems, often making their comparison difficult or even impossible. The fact that the Theory of Didactic Situations and the Anthropological Theory of the Didactic share their main assumptions and their 'research programme' (in Lakatos' terms) makes it easier to contrast them in the way each one reinterprets and reformulates the problems raised by the other. Starting with 'neighbouring approaches' thus appears as a sensitive way to approach the complexity of networking theories.

According to Rodríguez et al. (2008), we assume that any strategy to compare, contrast or network theories has to take into account the way theories question reality and formulate problems about it. This assumption leads us to consider as a *networking methodology* the comparison between the reformulations proposed by different theories of a research question raised by one of them. In this case, the question emerges from an empirical episode and a given set of data. We start this 'exercise' with the case of two theories close to each other, the Theory of Didactic Situations (TDS) and the Anthropological Theory of the Didactic (ATD). We first present the context where this study takes place, and then analyse the exchanges between the co-authors of this contribution around a particular research question, before entering a more general discussion about the potential of this methodology.

1. THE CONTEXT FOR THIS STUDY

This study is part of the work on the comparison of theoretical frames of a collective that emerged at CERME4, and whose first outcomes have been presented at CERME5 (Arzarello et al. 2008, Kidron et al. 2008, Prediguer 2008). Since CERME5, the group has orientated its work towards the development of networking methodologies. Different strategies are used for that purpose. One of these, which presents some similarity with the strategy used in the ReMath European project (Artigue 2007, Mariotti 2008), is the comparison between the formulations proposed by different theories when confronted to a given set of data and a research question raised by one of them. In our case, the research question emerged from the analysis of a video, which, from the very beginning, played a crucial role in the work of the group. It corresponds to a classroom session at grade 10 in Italy on the exploration of the properties of exponential functions in the Cabri-géomètre environment, and more precisely to the observation of a group of two students. In a first phase of the work, the different teams involved in the group analysed the video from their respective theoretical perspectives, what made clear that all of them, except the Italian colleagues, could not find what they needed for completing the analysis they aimed at in the information initially provided: the video and some documents about the classroom session. Each team was thus asked to make clear the kind of information it needed, and the demands of the different teams were discussed at a post-CERME5 meeting. One of the results of this discussion was a questionnaire to be answered by the teacher in charge of the class observed. When the extra information agreed upon, including the teacher's answers to the questionnaire, was disseminated, each team tried to complete its analysis, and the results were presented during a joint meeting in Barcelona. In their respective presentations, several teams referred to a particular answer made by the teacher, pointing out that, from their perspective, such an answer raised important and non trivial issues and deserved further discussion. The question and the answer were the following:

> *"During a lesson of this type, under what circumstances do you decide to get involved with a pair of students, and what kinds of things do you do?"*
>
> "I try to work in a zone of proximal development. The analysis of video and the attention we paid to gestures bring me to become aware of the so called 'semiotic game' that consists in using the same gestures as students but accompanying them with a more specific and precise language in relation to the language used by students. A semiotic game, if it is used with awareness, may be a very good tool to introduce students to institutional knowledge."

This episode of our collaborative work and the potential we soon suspected it could have if analysed in depth, was the source of the networking methodology we then developed. This methodology obeyed the following organization: the team working in TDS formulates a research problem using its own terminology; each team converts the problem according to its theoretical perspective; the team working in TDS comments on the new formulations, looking at the generic and specific issues; each team works on its specific question and reflects on the process followed.

In what follows we describe the exchanges that this methodology generated between the TDS and ATD perspectives, and analyse their networking potential.

2. EXCHANGES ON "SEMIOTIC GAMES"

2.1. A first perspective inspired by TDS

As mentioned above, a series of comments regarding the teacher's answer and the articulation of some precise questions was first elaborated within the TDS perspective by two of the co-authors of this contribution (MA & AL). We summarize the main lines of their argumentation below, the teacher being denoted by T.

First, MA & AL observe that the answer expresses the confidence that T has in the so called semiotic games to face a major didactic problem: the connection between, on the one hand, the mathematics produced by students in an adidactic situation, through the interaction with the adidactic *milieu* of this situation,[1] and on the other hand, the institutional mathematical knowledge aimed at. They add that this connection generally requires at least changes in the ways the mathematics at stake are expressed in order to progressively tune them with more conventional forms of expression; and that T obviously considers that he has a specific mediating role to play for making this connection possible and uses semiotic games as a tool for that purpose. In other terms, semiotic games can be considered as components of the praxeology (or more certainly of the different praxeologies) that T has developed in order to solve this didactic task. It is interesting to point out that this last sentence uses terms coming from ATD and not TDS, organizing a first bridge between them.

MA & AL then point out that this answer raises two interesting didactic issues:

The first one is that the situations proposed to students for building new mathematical knowledge do not necessarily have the adidactic potential that is necessary to enable the students to produce the mathematics to be produced under the constrained conditions of the classroom. What is achievable and achieved through an adidactic interaction with the *milieu* is often far from allowing the teacher to easily establish a meaningful connection with the mathematical knowledge aimed at. The discrepancy leads to different phenomena that have been discussed in TDS research (for instance Jourdain effects or "*dédoublements de situation*"), all the more as the teacher feels obliged to maintain the fiction that the mathematics knowledge he or she is expecting has to be produced by the students.

The second one is that the situations proposed to students for building new mathematical knowledge are very often what the TDS calls situations of action. They can lead to a linguistic activity but language issues are not their main concern. The characteristics of the *milieu*, the feedback available, do not make the productivity of the interaction with the *milieu* strongly dependent on the language used by the students. This is a fundamental difference with situations of communication often associated with the dialectics of formulation in the TDS.

Referring to their analysis of the video, MA &AL claim that the associated situations have a rich adidactic potential but also that this potential is a priori not sufficient to ensure the production of all the mathematical knowledge aimed at according to T's answers to the whole questionnaire. They also add that, even if the students have to produce narratives, the three situations which can be identified in the observed ses-

[1] The notion of *milieu* was introduced by Guy Brousseau as a main element of the Theory of Didactic Situations (Brousseau 1997). It refers to a system without any didactic intention that constitutes a key element of any adidactic situation. The reader unfamiliar with the TDS can find a very accessible introduction in Warfield (2006).

sion are closer to situations of action than to situations of formulation. They thus conclude that these situations constitute a priori good material to examine in context the potential and limits of semiotic games.

They also point out the specific status of T who is an expert teacher, but much more than that, due to his research engagement. According to them, this means that the confidence he expresses in the potential of semiotic games certainly has a solid experiential basis both in his personal practice and also in the practices of the research community he is involved in. Nevertheless, MA & AL's personal experience leads them to look at these semiotic games carefully, all the more when they are said to provide techniques for solving what are considered difficult didactic problems, and to try to understand under what conditions and why they can become efficient didactic techniques helping teachers face the difficulties described above.

The research question resulting from this analysis is the following:

How to identify characteristics of the semiotic game technique that would help us to understand its potential for:

- Compensating the possible limits of the interaction with the adidactic milieu to achieve the expected mathematical goals?
- Fostering the linguistic evolution linked to the needs of institutionalization processes?
- Identifying conditions required to activate this potential?

How to identify possible difficulties in the management of such semiotic games and possible effects of their possible malfunctioning?

2.2. Conversion of the research question within the ATD perspective

The answer to this question analysed below comes from the other two co-authors of this contribution (MB & JG) who work in the ATD perspective.

2.2.1. Some preliminary considerations

The ATD describes human practices (including doing mathematics and its teaching and learning) in terms of *praxeologies* composed by two complementary folds: a *praxis* or practical block (the "know-how") made of *types of tasks* and *techniques* to carry out these tasks; a *logos* or theoretical block (the "knowledge" in its narrow sense) that appears as an assemblage of discourses to describe, explain and justify the *praxis*.[2] The question formulated by MA & AL starts from a rather vague notion of 'semiotic game' that, in the ATD, can be considered as a *didactic technique* that T describes as follows: "the teacher starts using 'the same gestures as students but accompanying them with a more specific and precise language in relation to the language used by students". T's comments on the episode also reveal some *theoretical components* explaining and justifying the use of this technique, formulated in terms

[2] For the reader unfamiliar with the ATD, see Bosch & Gascón (2006).

of 'working in a zone of proximal development'. At the same time, the comments refer to a type of *teaching task* that is supposed to be performed with this technique: 'introducing students to institutional knowledge'. We are thus considering a *didactic praxeology* as it is evocated by the teacher.

Any 'didactic problem' (that is, a problem related to the teaching, learning, studying or diffusion of knowledge) can be generally identified both with a 'teaching problem' (that is, a question or difficulty that appears in the teacher's practice and that requires an appropriate *didactic praxeology*) and with a 'research problem' (that is, an open question for research in mathematics education). In both cases the problem is formulated in relation to a *teaching and learning process* and connected to a given mathematical content (which is a *mathematical praxeology* or a set of mathematical praxeologies). In this sense, the 'expected mathematical goal' that appears in the formulation of the question, as well as the 'proposed institutional knowledge' are mathematical praxeologies that can have different 'size': point, local, regional or even global.[3] According to Bosch & Gascón (2005), the ATD postulates that the minimal unit of analysis of didactic processes has to contain at least a *local* mathematical praxeology. Furthermore, this local level is considered as privileged or basic because, in order to be studied in an operative way, any didactic problem formulated beyond this level of analysis needs to be 'projected' into its local components. For MB & JG, in the ATD perspective, the initial research question can thus be situated in the very general problem of the study of *the conditions that make the building of local mathematical praxeologies in a given institution possible and the restrictions that hinder it*.

2.2.2. The dialectic media/milieu

At the beginning, the process of building local mathematical praxeologies can start from questions that arise within a point praxeology or in a small set of them. In any case, the driving force of the didactic process, what provokes the need to study or build a local praxeology integrating and completing the point praxeologies, is the emergence of questions that cannot be answered within the point praxeologies. How these questions arise in a given didactic process? What conditions are needed for a study community to 'take them seriously'? What '*media*' can help the study community to generate provisional answers and what '*milieu*' is available to test and modify these answers? These are still open questions and an in-depth analysis of what is called the 'dialectic media/milieu' seems essential to answer them.

[3] A *point* praxeology is generated by a unique type of tasks and is often characterized by a unique technique to deal with them; a *local* praxeology is generated by the integration of several point praxeologies within the same technology; a *regional* praxeology is obtained by coordinating, integrating or linking several local praxeologies through a common mathematical theory and a *global* praxeology is a connection of some regional praxeologies (Rodríguez *et al*. 2008).

According to Chevallard (2004), the elaboration of an answer to a real question supposes 'resources' or 'milieus'. In close connection with the TSD terminology, a 'milieu' is a system without any didactic intention in the interaction we can have with it during the study process. In this sense a milieu behaves as a fragment of 'nature'. Besides the notion of 'milieu', the ATD introduces the notion of 'media' as any system the main goal or intention of which is to supply information about a given issue. In any knowledge construction process a dialectics between a media providing new knowledge or information and a milieu able to give evidence of the validity of this information takes place. An extreme situation is when one takes the message coming from the *media* as it appears, without any need for testing it. The opposite side is the construction of knowledge from scratch, through only the confrontation with a *milieu*. The existence of a vigorous (and rigorous) dialectics between *media* and *milieus* appears to be a crucial condition for a study process not to be reduced to a simple copy of previously elaborated answers spread over different social institutions.

2.2.3. Formulation of the question in the ATD frame

(a) *'Semiotic games' and the limitations of the adidactic milieu*

The general didactic problem we are considering is the study of the didactic tools, devices or praxeologies that are necessary for the teacher to lead and for the students to carry out the process of building local praxeologies. With respect to the problem of the 'limits of the interaction with the adidactic milieu', it is important to notice that, from the perspective of the ATD, the dialectic media/milieu supposes that any milieu has limitations in the didactic process consisting in building a local praxeology as the progressive answer to a problematic initial question. Even if a given milieu can help contrast a partial answer to the initial question, it will always provoke the need of new *media* introducing new information having to be tested with new milieus, and so on. In this context, T's 'semiotic game' considered as a didactic technique, may be interpreted as a resource used by the teacher – acting as a 'media' – to supply students with praxeological components of the praxeology that is to be built.

(b) *Institutional didactic praxeologies underlying the 'semiotic games'*

Beyond the didactic techniques a given teacher can 'create', research in the ATD frame is interested in the *didactic techniques a given institution makes available* to the teacher and the students to manage the construction of mathematical praxeologies and, more particularly, to manage the media/milieu dialectics.

This institutional dimension is essential because it strongly determines the ecological conditions required by these didactic techniques to normally evolve in the considered institutions. More particularly, the existing institutional conditions influence the kind of *technical gestures* that can usually be made in the institution, as for instance the 'semiotic games'. Like any other didactic technique, 'semiotic games' need an institutionalised didactic *technology* to describe, justify, interpret and control their role in the didactic process. Beyond the technological level, it is also interesting to study what *theoretical* foundation supports this teaching technique and technology.

2.3. Back to the TSD and the formulation of new research questions

After the re-formulation of the research question raised by TDS, a second exchange took place between the teams. We extract from it what concerns the TDS and ATD perspectives. In their comments, MA & AL first point out that, considered as didactic techniques both in TDS and ATD, semiotic games are given two different functionalities according to the theoretical perspective chosen. They also suggest that from this situation can emerge interesting insights regarding the relationships between ATD and TDS:

> "According to ATD, a condition for a study process not to be reduced to a simple copy of previously elaborated answers is the existence of a strong dialectic between appropriate media and milieus. Such a theoretical position presupposes that any milieu has limitations in the didactic process consisting in building a local praxeology, a process which is seen as the progressive answer to an initial question. Within this approach, T's semiotic games find their place as a didactic technique used for the management of the media/milieu dialectic. We think that it will be interesting from this point of view to compare the vision that will be proposed concerning these semiotic games on the one hand by the ATD analysis projecting them in the media/milieu dialectic and on the other hand by TDS projecting them at the interface between adidactic and didactic processes. Having its origin in a theoretical context both distinct from ATD and TDS, it may provide a good opportunity for understanding better the similarities and differences between these two theoretical approaches regarding these crucial aspects."

Another element stressed by MA & AL is that the conversion of the initial questions within an ATD perspective makes a new dimension move from the periphery to the centre: the institutional dimension. ATD indeed obliges the researchers to consider that the study of any kind of didactic technique has to be situated within an institutional perspective. It cannot exist and develop without any institutional legitimation, any institutionalised didactic technology used to describe, justify, interpret and control its role in the didactic process. Within this perspective, what is of interest for research is clearly not the study of semiotic games as practices of individual teachers but the study of their institutional status and ecology, of their relationships with other institutional techniques available to teachers for managing the dialectics between media and milieus. MA & AL add that, in this particular case, the experimental status of the course to which the observed session belongs means that at least two institutions are involved and should be considered: the research institution and the high school institution.

Finally, the exchanges also make MA et AL reflect more globally on the first phase of the work, and the limitation of the perspective underlying it. The first phase consisted of using the TDS and ATD theoretical constructs to reflect about semiotic games, their didactic potential and limit, but the converse movement is also possible, leading to investigate what can be offered to TDS and ATD by having the ideas of semiotic game and the 'zone of proximal development' as functioning in T's 'practical theory' (Ruthven, 2006) entering the scene. This converse movement can also be insightful regarding relationships between TDS and ATD, and the possibilities of networking between them.

2.4. Main features of a didactic research problem

At this point of the networking between TDS and ATD, and in order to pursue the network with other theoretical frameworks, it seems necessary to locate the dialogue in a new position, more general and relatively neutral from an epistemological point of view. Three main features seem important to distinguish.

2.4.1. Institutional dimension of the didactic problems

In the ATD perspective, the expression 'semiotic game' appears as an element of the *teacher's didactic theoretical discourse:* it helps him interpret what happens in the classroom, take decisions, etc. In this sense, we are dealing with a component of the *spontaneous didactic praxeology* of a *concrete teacher*. A first difficulty appears concerning the personal or institutional dimension of this didactic praxeology.

Institutional praxeologies (and their ecology) are the ATD's primary object of study. To study them, we take as an empirical basis the personal manifestations of these praxeologies as well as their more collective or institutional manifestations: regular practices, discourses, texts, official documents, etc. The dialectic between *persons* and *institutions* can be made more explicit in the following terms. The institutions where praxeologies take place are composed of persons. Reciprocally, persons are always *subjects* of a complex of institutions and, as such, have a personal relation to praxeologies that can be explained to a great extent by the analysis of the institutional praxeologies they have encountered.

2.4.2. Mathematics as a core component of didactic problems

Taking into account the educational institutions' vision of teaching and learning processes is a basic methodological principle of the ATD. Otherwise, we run the risk of taking for granted the description of phenomena proposed by each institution – which can furthermore differ from one institution to another. More particularly, research on *didactic transposition processes* (Bosch & Gascón 2006) has shown the necessity for research to construct its own models of mathematical knowledge (or mathematical activities) in order to avoid taking for granted the models imposed by the dominant institutions. These models of mathematical knowledge should include the description of its construction, development and diffusion (and, thus, the mathematics teaching and learning processes).

2.4.3. The importance of the unit of analysis

Any essay to contrast or compare theories has to face a dilemma. On the one hand, to contrast theories, we need a 'common' empirical universe and, thus, we have to remain close to the educational institutions. On the other hand, each theoretical perspective constructs its own vision of this empirical universe, moving away from the educational institutions (to 'escape' from their dominant vision). This detachment is necessary in order to approach problems related to the teaching and learning of mathematics in a more operative way. However, it has always to maintain an accurate distance to the reality one wishes to study – and modify!

The notion of 'minimal unit of analysis' (section 2.2.1) appears as a basic aspect of the modelling of educational phenomena proposed by each theory. Starting from the way each perspective formulates MA & AL's question, we could make explicit what units of analysis are considered in each case and how they can be connected. This could be a good way to improve our capacity of describing and comparing not only the concrete research or practical problem formulated by each frame but also the type of problems that can be proposed and the kind of empirical data needed.

3. CONCLUSION

This contribution illustrates a methodology of 'networking theories' based on the study of a question, considering how the different research frameworks engaged in the networking can formulate and approach it, through a sequence of exchanges and progressive refinements. We have taken the interaction between TDS and ATD as a study case, considering two close frameworks that share the same scientific project. This proximity makes the networking easier because the discussion on the fundamental background of the theories can be avoided. It is important to recall that ATD emerged within the TDS, thus integrating the original research programme, its basic assumptions, the nature of the considered problems and phenomena and, more particularly, the need to question and model mathematical knowledge (that is, to take it as a specific object of study). Making this methodology productive with more distant approaches raises the necessity to make the basic assumptions of each one explicit and to contrast them. Another positive consequence of this methodology stems from the fact that the theories involved are questioned from an external construction, which in our case has given rise to two main contributions. The first one is the institutional dimension assigned (by the ATD) to the 'semiotic games' and the way it can be taken into account by the TDS. This issue has long been explored and largely discussed by research in both the TDS and ATD perspectives (Sensevy et al 2005). The second contribution is the comparison between the projections 'didactic – adidactic' and the 'media and milieu dialectics'. They emphasize an obvious difference in the way both theories take into account the milieu's insufficiencies and the changes in our relation to knowledge led by the technological evolution. It is important to note finally that, till now, little advantage has been taken from the inverse networking movement: considering the contributions made by the external perspectives to the development of our own one. For instance, by formulating problems which are not of first priority in our research programmes but the study of which can open unexpected lines of development. We finally postulate that making explicit the position adopted by research perspectives to the features considered in section 2.4 constitutes an essential step for the networking. This position is important because it delimitates what is considered a 'didactic research problem' and, consequently, contributes to characterise the object of study of our discipline.

REFERENCES

Artigue M. (2007). Digital technologies: a window on theoretical issues in mathematics education. In, D. Pitta-Oantazi & G. Philippou (eds), *Proceedings of CERME 5*, pp. 68-82. Cyprus University Editions.

Arzarello, F., Bosch, M., Gascón, J., Sabena, C. (2008). The ostensive dimension dimension through the lenses of two didactic apporaches. *ZDM The International Journal on Mathematics Education*, 40:2, 179-188.

Bosch, M., Gascón, J. (2005). La praxéologie comme unité d'analyse des processus didactiques. *In*: Mercier, A., Margolinas, C. (coord.), *Balises en Didactique des Mathématiques*, Grenoble: La Pensée Sauvage (pp. 107-122).

Bosch, M., Gascón, J. (2006). 25 years of the Didactic Transposition, *Bulletin of the International Commission on Mathematical Instruction* 58, 51-65, 2006.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht: Kluwer Academic Publishers.

Chevallard, Y. (2004). Vers une didactique de la codisciplinarité. Notes sur une nouvelle épistémologie scolaire. *Journées de didactique comparée* (Lyon, mai 2004). (http://yves.chevallard.free.fr)

Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M., Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *ZDM The International Journal on Mathematics Education*, 40-2: 247-264.

Mariotti M.A. (2008). Representing Mathematics with Digital Media. Deliverable http://remath.cti.gr.

Rodríguez, E., Bosch, M., Gascón, J. (2008). A networking method to compare theories: metacognition in problem solving reformulated within the Anthropological Theory of the Didactic, *ZDM Mathematics Education*, 40: 287-301.

Sensevy G., Schubauer-Leoni M.L., Mercier A., Ligozat F., Perrot G. (2005). An Attempt to Model the Teacher's Action in the Mathematics Class. *Educational Studies in Mathematics*, Vol. 59/1-3, 153-181.

Ruthven, K. (2006). Building practical theories for technology integration. *Journal of Science and Mathematics Education in Southeast Asia* 29(2), 97-121.

Warfield, V. M. (2006). *Invitation to Didactique.* Retrieved December 2, 2008 from http://www.math.washington.edu/~warfield/Inv to Did66 7-22-06.pdf.

COMPLEMENTARY NETWORKING: ENRICHING UNDERSTANDING

Ferdinando Arzarello (*), Angelika Bikner-Ahsbahs (°), Cristina Sabena (*)

(*) Dipartimento di Matematica, Università di Torino (Italia),
(°) Fachbereich Mathematik und Informatik, Universität Bremen (Deutschland)

Our analysis of data about one learning situation from two theoretical perspectives yields results that on the surface seem to be in conflict. Through networking of two theories we produce a fresh combined analysis tool, which deepens our understanding of the data in an integrated way. We elaborate this example to make explicit our two theoretical approaches and our networking strategies and methods.

INTRODUCTION

The goal of the paper is to show how networking different theories can help researches in entering more deeply into their research questions. More precisely, we will illustrate the limits of two theoretical approaches when used alone to analyse a classroom teaching situation, and the benefits of networking. As a result, data analysis and learning processes understanding is strongly enriched.

The main question faced in our research concerns how mathematical knowledge about the growth of the exponential function is achieved in a specific socially supported learning processes. This requires properly defining the objects of our research, the method and the tools for observation (Prediger et al., 2008). As to the objects, we distinguish two deeply linked components: the social interaction among the subjects, and the epistemic issues in such learning processes.

Our networking strategy is worked out through analyses of empirical data. The same teacher-student-interaction is analysed from two theoretical perspectives that on the surface seem to be in conflict: the *interest-dense situation* and the *semiotic bundle* analysis. Using the former, it appears that the thought process of a student is disturbed by the social interaction with the teacher. However, no disturbances appear using the latter. We will show that through adding an epistemological perspective this conflict can be cleared away since the results can be integrated into a common view deepening our insight from both theoretical perspectives. This experience will be a starting point for a case of local integration of the two theoretical perspectives and some methodological reflection concerning networking strategies and methods.

ADOPTING TWO DIFFERENT PERSPECTIVES

Interest-dense situations and its epistemic process

So called interest-dense situations (Bikner-Ahsbahs, 2003) are those in which a maths class shows interest in the mathematical topic or activity, they occasionally occur within discursive processes in everyday maths lessons. In these situations the students become deeply involved in the mathematical activity, deepen their mathematical insight constructing further reaching mathematical meanings and begin to appreciate the mathematics they learn. To achieve some mathematical knowledge the students activate epistemic actions (actions that are executed in order to come to know more). Through social interactions the class collectively coordinates the epistemic process. In this way collective epistemic actions are constituted by social interaction. In contrast to non interest-dense situations, all interest-dense situations lead to the epistemic action of structure seeing (perceiving a mathematical pattern or rule referring to an unlimited number of examples).

The genesis of interest-dense situations is supported by a special kind of social interactions: The students are driven by their own way of thinking. They follow their own questions and ideas about the mathematical object that they want to know more about. In this case the students' actions are independent of the teacher's expectations. In interest-dense situations the teacher's expectations do not control the situation. Rather the teacher focuses on supporting the students' thinking. If the teacher's behaviour is controlled by his own expectations the emergence of an interest-dense situation is interrupted, and the learning process is disturbed (Bikner-Ahsbahs, 2003).

The ways in which the teacher and students socially interact can be analysed on the three levels (Davis, 1980; Beck & Meyer, 1994). Speaking, a person expresses something on three different levels. On the locutionary level he/she says something, on the illocutionary level, he/she tells something through the way of saying something. The perlocutionary level is concerned with effects: "a speaker saying something produces an effect on feelings, thoughts, or actions of the audience, other persons, or himself" (Davis (referring to Austin and Searle), 1980, p. 38). In our example, G locutionarily says: "for a very big variable *a*, when the exponential function ($f(x) = a^x$) and this straight line (which he assumes), meet each other, it (meaning the straight line) approximates the function very well because..." being interrupted by the teacher's request: "what straight line, sorry?". By using broken language, G tells the teacher that (illocutionarily) he is working out his train of thought while speaking. Starting the sentence with "because", he indicates on the illocutionary level that his train of thought is not yet finished. On the perlocutionary level we observe an effect; the teacher's request. In order to comprehend how the epistemic process in a discursive learning situation is socially supported or hindered; the analysis of social interactions is done on these different levels and is complemented by an analysis of the epistemic process. The term "non-locutionary level" will embrace the illocutionary and perlocutionary level.

The semiotic bundle perspective

The semiotic bundle perspective lies on two basic assumptions:

- the teaching-learning process inherently involves *resources of different kinds*, in a deep integrated way: words (orally or in written form); extra-linguistic modes of expression (gestures, glances, …); different types of inscriptions (drawings, sketches, graphs, ...); different instruments (from the pencil to the most sophisticated ICT devices), and so on (for some examples see Arzarello, 2006);

- such resources may play the role of signs (according to Peirce's definition[1]) and therefore can be considered as *semiotic resources*.

Differently from other semiotic approaches, the semiotic bundle construct allows us to theoretically frame gestures and more generally all the bodily means of expression, as semiotic resources in learning processes, and to look at their relationship with the traditionally studied semiotic systems (e.g. written mathematical symbolism):

> "A semiotic bundle is a *system of signs* — with Peirce's comprehensive notion of sign — that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher" (Arzarello et al., in print).

In teaching-learning contexts the different semiotic resources are used with great flexibility: the same subject can exploit simultaneously many of them, and sometimes they are shared by the students and by the teacher. All such resources, with the actions and productions they support, are important for grasping mathematical ideas, because they help to bridge the gap between the worldly experience and the time-less and context-less sentences of mathematics. An interesting phenomenon that has been identified within such an approach is the so called *semiotic game* (Arzarello, 2006; Arzarello et al., in print). A semiotic game happens in the teacher-students interaction when the teacher tunes with the students' semiotic resources and uses them to guide the evolution of mathematical meanings. We have analysed various examples in which the teacher repeats a student's gesture, and correlates it with a new term or with the correct explication given using natural language and mathematical symbolism (ibid.). Semiotic games constitute therefore an important strategy in the process of appropriation of the culturally shared meaning of signs.

An example analysed from the two perspectives

In this example, students (grade 10 of a scientific oriented high school) are working in pair on an exploratory activity on the exponential function. They are using a dynamic geometry software to explore the graphs of $y = a^x$ and of its tangent line[2] (a is a parameter whose value can be changed in a sliding bar). At a certain point the teacher has asked the students the following question: what happens to the exponential function for very big x? We propose a short excerpt from the interaction between the teacher and one pair of students (G and C) about this question.

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[1] As *sign* or *semiotic resource*, we consider anything that "stands to somebody for something in some respect or capacity" (Peirce, 1931-1958, vol. 2, paragraph 228).

[2] The line is actually a secant line; the secant points are so near that the line appears on the screen as tangent to the graph. This issue has been discussed in the classroom in a previous lesson.

1 [00:00] G: but always for *a* very big this straight line (pointing at the screen), when they meet each others, there it is again…that is it approximates the, the function very well, because…

2 T: what straight line, sorry?

3 G: this …(pointing at the screen) this, for *x* very, very

(00:14) big

00:14 G: the hand goes upwards

4 T (00:16): will they meet each other (00:17)? [suggestive connotation in the sense of "do you really think so?"]

00:16 T: pointing two forefingers

00:17 T: crossing the two pointed forefingers

00:19 G: two forefingers touching each other

5 G: that is [cioè][3], yes, yes they meet each other (00:19)

6 T: but after their meeting, what happens?

7 G: eh..eh, eh no, it make so (00:24)

00:24 G crosses the left hand over the right one; T is keeping the previous gesture

8 T: ah, ok, this then continues (00:27), this, the vertical straight line (00:28), has a well fixed *x*, hasn't it? The exponential function later goes on increasing the *x*, doesn't it (00:31)? Do you agree? Or not?

00:27 T moving rightwards his left hand

00:28 T: right hand vertically raised

00:31 T: moving rightwards his right hand

[3] The expression "cioè" in Italian means literally "that is". Over-used by teenagers, it introduces a reformulation of what just said. As it is likely in this case, it can have the connotation of "I am sorry but".

9 G: yes […]

10 T (addressing C): He [G] was saying that this vertical straight line (pointing at the screen) approximates very well (00:43) the exponential function

11 G: that is, but for very big *x* (00:46)

12 T: and for how big *x*? 100 billions? (00:51) *x* = 100 billions?

00:43 T raises both hands

00:46 G moved his left hand high wards

[00:51 T: raised his hand and keeps it still]

13 G: that is, at a certain point…that is if the function (00:57) increases more and more, more and more (00:59) then it also becomes almost a vertical straight line (1:03)

00:57 G raises his left hand

00:59 G moves his hand upwards

01:03 G's hand is vertical

14 T: eh, this is what seems to you by looking at; but you have here *x* = 100 billions (01:08), is this barrier overcome sooner or later, or not?

01:08 T: keeps his right hand in the vertical position

15 G: yes

16 T: in the moment it is overcome (01:12), this *x* 100 billions (01:13), how many *x* do you have at disposal, after 100 billions? (01:14)

01:12 T crosses left forefinger over right hand

01:13 T raises his right hand

01:14 T moves right hand rightwards, repeatedly

17 G: infinite

18 T: infinite… and how much can you go ahead after 100 billion?

19 G: infinite points

20 T: then the exponential function goes ahead for his own business, doesn't it? [01:26]

The analysis from the perspective of interest-dense situations

How is the emergence of an interest-dense situation supported or hindered? In line 1 G begins to construct mathematical meanings about the growth of the exponential function in broken language as described above. In this moment the teacher interrupts him: Apologising, the teacher illocutionarily indicates that he normally would not interrupt the student, but in this case an interruption is necessary. The teacher perlocutionarily might want G to feel accepted, however, saying sorry indicates also that there is something wrong with the "straight line". Locutionarily the teacher says: 'tell me what straight line you mean'. However, G does not react on the locutionary level; he describes the condition for his explanation in line 1: "for very big *x*"; just as he was asked to do in the task. The teacher's question "They will meet each other?" is (illocutionarily) posed in a suggestive way. Perlocutionarily, the teacher wants to get the answer: 'no, they don't meet'. However, G withstands the teacher's demand and answers *that they meet* (5). This is supported through adopting the teacher's finger crossing gesture (6, 7). On the locutionary level, we would see only the question and the answer. On the non-locutionary levels there is negotiation underneath. Looking only at the lines 1 to 5, an interest-dense situation is about to emerge. From the theory of interest dense-situation we could predict how the teacher could support or hinder the emergence of interest-density. Focussing on the student's ideas he would support it, acting according to his own thinking process or his expectations he would interrupt the emergence of it.

In the sentence that follows, the teacher starts to build up an argumentation as a proof of contradiction following his own train of thought and not that of the student. In line 8, he constitutes his base of argumentation. In order to include G into the process, his rhetorical questions "do you agree? Or not?" demands G's agreement. Summarising G's statement from line 1 grammatically more precise (10), the teacher establishes the statement that he wants to prove being false. G's modification "but for very big *x*" locutionarily looks like a complementary argument, but illocutionarily he corrects the teacher. G only partially agrees, because his description was based on 'very big *x*' (11). Again, G indicates that his train of thought is a bit different. Perlocutionarily G succeeds at this moment because the teacher changes his focus; locutionarily taking up the student's idea in the question: "for how big *x*?" (12). G seems to feel encouraged to explain: "that is, normally does not arrive at a certain point, the function increases more and always more, then still it becomes almost a vertical straight line …". Again, an interest-dense situation is about to begin. Then, on the non-locutionary level, the teacher expresses understanding G's view (14). However, through saying that, he also says that the student's way of arguing is false. He proves this by a proof of contradiction which he closes by the rhetorical question: "or not?" After the proof, G gives up to follow his own train of thought. The emergence of interest-density dries up.

Semiotic-bundle analysis

We see both student and teacher enacting a semiotic bundle composed by words, gestures, and inscriptions on the screen of the laptop. The basic point of discussion regards the behaviour of the exponential function for big base *a* and big values *x*. G thinks that in this case, the function can be approximated by a vertical line (#1-3). Such a conjecture is fostered by the image from the dynamic geometry software the students are using (see Figure 1): the tangent line appears in fact as almost vertical, and the exponential function comes to be perceptually confused in it. The teacher wants to clarify whether the student is thinking to a vertical asymptote (#4-6). Asking about an hypothetic meeting of the function with the straight line, he is representing the graphs by means of his iconic gesture (00:17): his right forefinger stands for a vertical line, and his left forefinger is inclined to represent the exponential function graph. G (#5-7, 00:19 and 00:24) is tuning with the teacher's semiotic resources, both speech and gesture. With his hand, he represents the graph of the exponential crossing the vertical line (00:24): he is answering the teacher's question by means of the gesture. The teacher (#8) accepts such an answer and endeavours in making explicit the idea that the domain of the exponential function is not limited, and therefore its graph intersects any vertical line. To do so, he uses both speech and gestures (see #8-20, and the related pictures). Let us enter into the dynamics of the semiotic bundle. In order to include C in the discussion, the teacher reports G's observation. By repeating G's words (#10) he is tuning with the student's semiotic resource (speech). But through gestures (00:43, 01:12, 01:13), he is making explicit the behaviour of the exponential function, i.e. the fact that it crosses any vertical line. The teacher is showing what we call a semiotic game, in that he is tuning with the student's semiotic resource, and is using another resource to make meanings evolve towards mathematical ones. The gesture appears a powerful resource, since it allows him to refer to what cannot be seen in the representation on the screen, and that is still difficult for the students to be conveyed in speech. In particular, gesture seems a suitable means to refer to very big values and to evoke their infinite quantity (01:14). If we now turn to G, we see that he does not appear to have profited from the teacher's semiotic game. Let us focus on lines 11-13 and related pictures. In his words we can see that he is still insisting on the idea that the function will become "almost a vertical straight line", but above all his gestures appear very different from the teacher's ones. In fact, whereas the teacher's gestures link big values of *x* with the right location in space (hand moving rightwards: 00:31, 00:51 and 01:14), the student's ones link big values of *x* to top location in space (hand moving upwards (00:46, 00:57, 00:59 and 01:03). From a cognitive point of view, they are

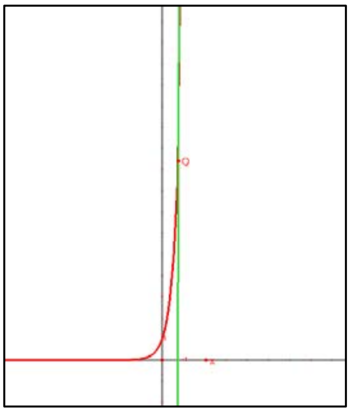


Figure 1

adopting different metaphorical references and only the teacher's one is consistent with mathematical signs (i.e. the Cartesian plane).

AN EMPIRICALLY BASED INTEGRATION

Based on the theoretical account and the empirical analysis, we can consider the two theories as *complementary*: they shed light on different aspects of the teacher-students interaction. However, by using the two theoretical lenses separately it appears that there is something important missing in each case. The strength of the interest-dense situations perspective is the possibility to predict their emergence according to the type of social interactions that hinder or foster it. In fact it includes the analysis of the locutionary and non-locutionary levels of speech and shows negotiations underneath the content. This approach is able to describe how the epistemic process proceeds and provides deeper insights into the social interaction process that foster or hinder the emergence of interest-dense situations, including structure seeing. However, the student and the teacher are not able to merge their argumentations although there is a lot of negotiation about whose train of thought will be followed. Neither the teacher nor the student is able to engage with the other's perspective. The analysis shows a gap that cannot be overcome, but is unable to give the tool to find out why this is so. By looking at a wide range of signs (in Peirce's sense), the semiotic bundle analysis identifies the semiotic game between teacher and student, and allows the game to be properly described. However the theory is not able to fully explain the reason why the student does not gain much from such semiotic game. In most other cases we had observed that the students succeeded to learn through semiotic games (e.g. see Arzarello et al., in print). One difference that can be identified within the theoretical frame is that this time the semiotic game applies the gesture-speech resources in reverse way with respect to semiotic games analysed as "successful". In this case, in fact, the teacher tunes with students' speech and uses gesture to foster meaning development; in other cases (see Arzarello et al., in print) it was the other way round: tuning with gestures and fostering meanings through words. We could conjecture that the characteristics of gestures as semiotic resource are not apt to this kind of didactical support, and indeed this can be a research problem to investigate. But within the semiotic bundle theory we are not able to say why such semiotic game did not work. The discussion so far leads us to argue that the simple juxtaposition of the two perspectives is not enough to deeply understand what's going wrong in the analysed episode. To go a step further, we start from the example to combine and locally integrate the two theories. The *combination* provides a tool to investigate how each sign of the semiotic bundle may contribute to the locutionary or non-locutionary aspects of the interaction. For instance, a gesture can support locutionary as well as non-locutionary features that play important roles in the interaction (see Figure 2). In the episode, gestures illustrated in pictures 00:19 and 00:24 at the locutionary level show the behaviour of the graph in iconic way, and at the non-locutionary they show that the student is trying to agree with the teacher's perspective. The hands in fact are used in the same configuration as the teacher (observe the teacher in the same pictures); in the entire episode this is the only case in which it happens. In all the other cases, G's gestures have very different configurations. Concerning the words, a similar situation is constituted; at the locutionary level G's words affiliate to the teacher's perspective. But at the non-locutionary levels the teacher and G do not fully agree with each other using words.

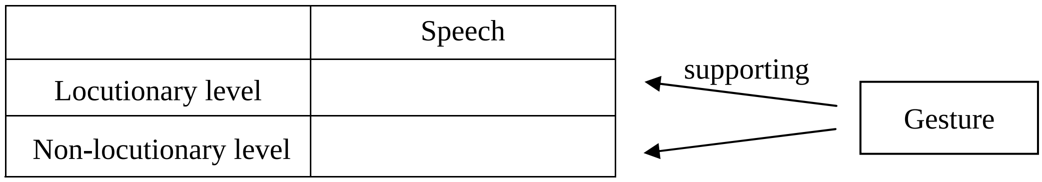


Figure 2: Two-level-analysis of semiotic resources

With the aim to answer the question what exactly did not work in the student-teacher interaction of the episode, we propose an integration of the two combined theories adding an *epistemological* dimension to the analysis above; that means to carefully consider the epistemological points of view of the teacher and of the students. By *epistemological points of view* we mean the background of the piece of knowledge that a subject thinks can give sense to a specific situation. The epistemological point of view is not always explicit: it appears not only from the locutionary dimension of the semiotic resources used by a subject but also from the non-locutionary ones. Moreover, it can be partially revealed by the epistemic actions produced by the subject. Of course the epistemological point of view with respect to a situation can vary with the subjects. For example, that of a student can be different from that of the teacher or of another student. But this difference might not be apparent although the dynamics of a didactic situation in the classroom might be deeply influenced by it, especially when the teacher is not aware of it or does not take into account the epistemological points of view of his students. This is exactly what happened in the episode analysed above. We observe a semiotic game articulated in a tuning in words and a dissonance in gestures: the teacher is repeating G's words (#11-12), but he is performing completely different gestures (see, that in 00:46 G's hand is moving upwards, to indicate big values, whereas in 00:51 the teacher's hand is moving rightwards). The dissonance in gesture is a signal that the teacher and the student are showing different points of view: the teacher relies on a formal theory (Weierstrass definition of limit) using potential infinite; the student relies on his perception imagining what happens "for very big *x*" (#11). It is not so clear what the student means: possibly he has been influenced by perceptive facts (see the discussion above) and perhaps he is thinking within an "actual infinite" perspective, even if this point is not so explicit here. The analysis of the semiotic game including the epistemological dimension allows us therefore to say that there is an *epistemological gap* between teacher and student, and to hypothesise that this gap prevents the teacher from suitably coaching the student's knowledge evolution and the student from profiting by the interaction with the teacher. Therefore the emergence of an interest-dense situation was not successful.

CONCLUSIONS

Presenting an empirical case of networking of theories, we showed that through a local integration two theoretical approaches can be enriched (Prediger et al., 2008). This was possible because the theories provided two complementary observation tools: one at the level of discourse analysis describes social interactions and their epistemic processes; the other at the level of gesture analysis describes learning from a semiotic perspective. The starting point of the theoretical integration was based on the empirical data analysis whose meaning was not clarified by any of the two theories. This stall was overcome by suitably combining the two approaches: adding an epistemological dimension made possible to locally integrate the two theories, so uncovering blind spots in both.

The results of our analysis could have important didactical consequences: in fact from them it seems possible to design a fresh role for the teacher in supporting students' learning processes. According to the combined analysis of the semiotic and linguistic features, integrated with the epistemological dimension, the teacher could develop suitable interventions, taking care both of the social interaction and of the epistemological issues with the help of semiotic resources.

REFERENCES

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latino americana de Investigación en Matemática Educativa*, Vol. Especial, 267-299.

Arzarello, F., Paola, D. Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97-109.

Beck, C. & Meyer, H. (1994). Mathematikdidaktik als Textwissenschaft [Mathematics education as the science of text]. *Journal für Mathematikdidaktik (JMD)*, 15 (1/2), 35-78.

Bikner-Ahsbahs, A. (2003). A social extension of a psychological interest theory. In N. A. Peitman, B. J. Dougherty & J. T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PME-NA*, Vol. 2, pp. 97-104. Honululu, HI: University of Hawai'i. College of Education.

Davis, St. (1980). Perlocutions. In J. R. Searle, F. Kiefer & M. Bierwisch (Eds.), *Speech act theory and pragmatics* (pp. 37-55). Dordrecht, London: D. Reidel Publishing Company.

Peirce, C.S. (1931-1958). *Collected Papers* (Vol. I-VIII). In C. Hartshorne, P. Weiss, A. Burks (Eds.). Cambridge, Massachusetts: Harvard University Press.

Prediger, S., Bikner-Ahsbahs, A. & Arzarello, F. (2008). Networking strategies and methods for connection theoretical approaches: first steps towards a conceptual framework. *ZDM-International Journal on Mathematics Education*, 40(2), 165-178.

INTERPRETING STUDENTS' REASONING THROUGH THE LENS OF TWO DIFFERENT LANGUAGES OF DESCRIPTION: INTEGRATION OR JUXTAPOSITION?

Christer Bergsten, Eva Jablonka

Linköping University, Luleå University of Technology

This contribution exemplifies the interpretation of a common set of data by using two languages of description originating from different theoretical perspectives. One account uses categories from a psychological and the other from a sociological perspective. The interpretations result in different explanations for the students' struggles with sense making. However, the results cannot be integrated into a combined insight, but only be juxtaposed.

INTRODUCTION

The role of theory in mathematics education research has many facets so that comparisons of outcomes of research carried out within different perspectives remain a challenging and complex task (Silver & Herbst, 2007; Radford, 2008). The observed diversity of theories, paradigms, and frameworks in the field has called for serious efforts of understanding, comparing, contrasting, coordinating, combining, synthesising, or integrating different perspectives (Prediger, Bikner-Ahsbahs, & Arzarello, 2008). In line with this work, this paper, by way of an example, sets out the task to construct two *accounts of* a transcript from a video taped problem solving session for the purpose of comparing and contrasting different *accounts for* it (Mason, 2002), based on two languages of description stemming from two different theoretical traditions. In the session pairs of students were working on tasks on limits of functions, a topic where most of the research about students' sense making has been done from a cognitive psychology approach (Artigue, Batanero, & Kent, 2007). For an alternative account, we have chosen a sociological approach, which is rather uncommon but has the potential of overcoming deficit orientated interpretations of students' struggles.

Much of the research that aims at accounting for the problems students have, focuses on a distinction between "intuitive" and "formal conceptions" of limits (e.g. Harel and Trgalova, 1996, pp. 682-686). The notion of limits of functions is conceived as one where intuitive conceptions of infinity may prove insufficient or even contradictory to a formal mathematical treatment (Núñez et al, 1999). As an exemplary of approaches that account for students' problems with limits of functions in terms of the individual's cognition, we produce an account of the data that draws on the work of Alcock and Simpson (2004, 2005). Their conceptualisation describes an interplay between modes of representations and beliefs about oneself and the role of algebra in reasoning about limits.

Starting from a sociological perspective, in a second attempt, we outline an account of the students' productions in terms of the dilemma they face when participating in different types of discourses. This interpretation draws on a language of description developed in the context of studies of recontextualisation that represent a structuralist tradition (Bernstein, 1996). In drawing on Bernsteins's theory, a successful student can be described as being able to realize in which context she participates and produces what is expected in this context, that is, the student must have access to "recognition rules" and "realisation rules" in order to produce "legitimate text". The ultimate agenda of such an approach is to explain how the students' access to these rules is distributed unevenly with respect to their different backgrounds. For our account of the empirical text from the problem solving sessions, we use categories of expression and content of mathematical problems from the perspective of recontextualisation of different types of discourses about limits of functions.

THE INTERVIEW SITUATION

Six beginning engineering students from a first semester calculus course volunteered to participate in the video study, where they were working in pairs to solve problems on limits of functions. Each session lasted for about 45 minutes. After an introductory question about the concept of a limit and its definition, the students were asked to investigate the limits of functions. The type of problems chosen were similar to the ones they encountered in the course: to find the limits as $x \to \infty$ and as $x \to 0$ for the three functions $f(x) = \frac{2x}{x^2 + \sin x}$, $g(x) = \frac{1}{x} - \frac{1}{x^2}$, and $h(x) = \frac{\ln(1+x^2)}{x}$.

For our accounts presented below, we used the transcribed protocol from the work of two pairs (A and B) of students on the function $h(x)$ and on the introductory question.

At the time of the interview the lectures had covered the definitions and basic properties of limits and continuity, and introduced and proved theorems about standard limits such as $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$, as well as worked examples. The textbook provided an exposition of an introductory calculus course based on the standard $\varepsilon - \delta$ definition of limits and continuity. In particular, standard limits were proved within this theory and used as theoretical tools to investigate the limits of functions given in algebraic form. Other techniques taught include removing dominating factors, extension by the conjugate expression, and change of variable. The approach was algebraic and non-numerical. Occasionally, diagrams were used. The teacher of the course sets out his agenda as follows (see Bergsten, 2007, p. 63):

> I want to present, to make things seem true, the most important I think is that students believe they understand better what a concept means. To exemplify what you can handle practically, to illustrate the standard way of doing things.

In the lecture the teacher made some efforts to integrate formal algebraic treatment with non-formal ideas about limits and behaviour of elementary functions (ibid.).

ACCOUNT 1: INDIVIDUALS' BELIEFS AND PREFERENCES

The style of work of Anne and Adam is dominated by algebraic manipulations across all tasks, where the observed notations are used mainly as keys for performing procedures that hopefully will lead to a possibility to apply a standard limit. This is done immediately when starting a new task, without prior discussion about how to attack the problem or what can be "seen" by considering properties of the functions involved. In the transcript, when discussing the case where *x* tends to infinity for the function *h*(*x*), Anne immediately suggests making a change of variables:

Anne:	Change of variables.
Adam:	...yes ... I think you get ... the logarithm can be rewritten, the function inside.
Anne:	No, we can't touch the function inside [writes, Adam looks at her seemingly puzzled] there is no expression for LN X plus LN Y equal to LN X plus Y.
Adam:	Yes yes but you can write it as LN one plus X ... that part [points] one plus X square can be written as ... one plus ... one minus X.
Anne:	Yes, equal to LN [inaudible, Anne writes].
Adam:	It does not help much in this case.
Anne:	No [erases what she wrote].

While solving this task no diagram is drawn or point made on properties of the functions involved that could lead the process forward. Standard limits and comparison tables are recalled as incitements and as clues to continued algebraic manipulation. Uncertainty in recalling these facts correctly does not prevent them from proceeding the algebraic explorations, possibly thinking it will eventually lead to a result:

Anne:	I must elaborate further on that one and see if it works.

The work goes on along the same lines in all tasks, trying to remember what one can do and trying out different algebraic methods, sometimes ending up in what could be called an algebraic mess, using expressions like "this is just impossible". In the following excerpt the students substitute $1+x^2$ by *t*.

Adam:	If we in the original expression extend with ... the square root of minus one ... T minus one in the denominator, LN T the square root of T minus one ... that one was not much better [looks at Anne].
Anne:	[writing] This is also unnecessary because we can't do this, it is the same shit ... doesn't matter ... than we have that this one moves this one moves and then this one moves.
Adam:	Yes all tend to infinity.
Anne:	To be honest, I think that infinity is the answer, as ... when I changed variables.

The last sentence indicates a weak "internal authority", as she cannot find a method that works, and on another occasion (on problem *f*) this is directly expressed:

Anne:	The question is if it is correct. Now I just want to know the right answer.
Interviewer:	You don't feel confident with the result?
Adam:	I can't say it should be another result, but this is a kind of task where I feel I could easily make a mistake.
Anne:	Yes, me too.
Adam:	By some change of variable it can be possible to make it tend to zero. /…/
Anne:	I think it is zero in both cases. What was the answer?

This predominantly algebraic way of working seems to be in contrast to the response to the opening question on the meaning of a limit, where they initially describe it verbally as a dynamic process using words like "approaching" but then prefer to make a drawing and add gestures when talking about it. However, as these images do not seem to have a link to their subsequent work on the problems they may lack a sufficient generality to justify their reasoning (cf. Alcock & Simpson, 2004).

Also the students in pair B describe the mathematical notion of limit as a dynamic process of 'approaching' but seem to accept both a potential and actual infinity, as when they discuss the arrows commonly used to denote limits:

Bob:	Yes I would maybe miss a little arrow ...
Ben:	Yes.
Bob:	... in front of A [i.e. the limit], tends to A, but I don't know if ...
Ben:	it gets so very close, yes goes to A.
Bob:	Yes, you usually don't have those arrows like that. But the function attains the value A when X is infinitely large, is a very very large number, don't know if I need to add more.
Interviewer:	Do you agree?
Ben:	Yes.

They also state that it is more easy to explain when using a diagram. However, their diagram is more elaborated and seems to support their thinking during the work with the problems. For pair B this work proceeds in quite a different manner from pair A, dominated by more informal reasoning about the size of the quantities of the different parts of the given functions. They frequently use the expressions "a very small number" and "a very large number". In 'simple' cases this way of reasoning is functional but in the case $\lim_{x \to \infty} h(x)$, this kind of intuitive method proves insufficient to find the limit even after 15 minutes of work:

Bob:	Zero times infinity is ok, almost zero times infinity is more tricky, it is not really zero but only tends to it. So it can be almost anything. Do we get anywhere? [looking at Ben]
Ben:	No [Bob laughing].
Bob:	Yes, but which one goes more, does that one go more to zero than that one to infinity? No it goes more to infinity than to zero, I think. [silence]

It seems as if algebraic methods, shown in the lectures, here are tried only when the conceptual approach does not produce an answer. However, when it does these students do not feel any need to verify the solution formally by the use of proven theorems on standard limits. They rely on "internal authority".

Internal authority is also evident by the use of the words "I think we are done" in the case $\lim_{x\to 0} h(x)$, after identifying a standard limit and applying it after expanding the term by x. But again no algebraic manipulations are performed on $\lim_{x\to 0} g(x)$, where they reason about approaching zero from the right or from the left. They conclude, after testing a numerical value, drawing a diagram and comparing infinities, that $g(x)$ tends to negative infinity. However, Bob is not fully satisfied:

Ben: So this [i.e. when approaching zero from the right] must also be negative infinity, don't you think so?

Bob: Yes, but it is kind of delicate when you take infinity minus infinity, it is kind of vague. But if we accept this way of reasoning with infinities of different size, then we have found that, if it is correct.

Thus, relying on internal authority might have prompted questioning the bases of their arguments and imply an uncertainty about the correctness of the result.

ACCOUNT 2: WEAKLY / STRONGLY INSTITUTIONALISED DISCOURSE

For the purpose of analyzing the recontextualisation of domestic practices in school mathematics texts, Dowling (2007) introduces a "relational space" of domains of action that differentiates between content and expression of a text, both being weakly or strongly institutionalised (see Table 1). Esoteric domain text refers to the conventional institutionalised mathematical language and its strongly classified specific meanings. In descriptive domain text, the expression is conventional mathematical language though its object of reference is not institutionalised mathematics. In expressive domain text, a mathematical concept or procedure etc. is expressed via signifiers that are not or weakly institutionalised (in an extreme case via non-mathematical signifiers). Public domain text is text with both weakly institutionalised forms of expressions and content.

The following interpretation employs these notions. As the context is a university lecture in calculus, public domain text cannot be expected to be found. The oral discourse in the lecture analysed in Bergsten (2007) included metaphorical language

	Content (signifieds)	
Expression (signifiers)	strong institutionalisation	weak institutionalisation
strong institutionalisation	*esoteric domain*	*descriptive domain*
weak institutionalisation	*expressive domain*	*public domain*

Table 1: Domains of Action (Dowling, 2007, p. 5; layout adjusted)

and gestures describing graphs of functions in terms of motion and direction as well as hints about what to do when applying standard procedures.

The written discourse focused on algebraic representations. The topics were presented with very detailed formalisations, very much in line with the textbook (co-authored by the lecturer), that is, as esoteric domain text drawing on strongly classified and institutionalised language and meanings. So it is the oral discourse that is situated in another domain, a domain of visuo-spatial and movement metaphors that are used for describing the Cartesian graphs, “the behaviour”, of functions and their limits (in terms of shape, growth, getting bigger and smaller and approaching). The meanings in this discourse are weakly classified, as are the modes of expressions. In the course of establishing the esoteric discourse, this discourse is re-contextualised from the perspective of an algebra of functions and their limits, and in doing so the first is subordinated to the latter. The students attempts to solve the tasks in the interview situation can be interpreted as a struggle to produce a legitimate text, that is an esoteric text. However, if they discussed with their peers and approached the solutions in terms of the weakly classified oral discourse, they were faced with a problem of recontextualisation. However, in the introductory question of the interview, they were asked to explain the concept of limit, which is a quite different challenge. The interviewer shows to the students a piece of technical language from the course: “$\lim_{x\to\infty} f(x) = A$” and asks:

Interviewer: Imagine you have a friend who just started such a course in calculus and has never seen this. How would you explain to him what this means?

The students are faced with the problem to recognize what a legitimate text in this interview situation would be. Into which domain has the expression to be translated for this imaginary friend?

Anne and Adam interpret this question as a task to produce expressive domain text. They first have to establish this new domain and start negotiating the translation and eventually agree that this new domain includes drawings of examples of functions. The technical language comprises “x”, “function”, “A” (which remains untranslated), “LN-function”. The expression “$\lim_{x\to\infty} f(x)$” is translated into “the limit”.

Anne: This is an expression for the limit. One looks at how a function behaves when X tends to infinity…and when X tends to infinity and the function approaches a constant which is called capital A, so it is convergent, as one calls it. This means that one can say the function then approaches a value if it does not go on …

Adam: It approaches a finite value then, so it is bounded, a bounded function.

Anne: This is a little hard without drawing it.

Adam: Yes, this is hard to explain, it is more easily explained with a figure, I think.

After two comments of Adam who talks about the value going “closer and closer”, the interviewer interferes by asking them whether they would want “to draw a figure for that friend”:

Anne: I think the friend should get a clearer picture in any case [Adam draws quietly, Anne watches] … yes [approves the figure and holds up the paper to the friend and smiles].

Adam: This is a function that approaches but never really reaches [illustrates with a gesture].

Anne: A bit like LN one can say

Adam: Yes, LN-function.

Anne: This looks like an LN [both laugh].

In their conversation while solving the tasks $\lim_{x\to 0} h(x)$ and $\lim_{x\to\infty} h(x)$, they focus on associating it with a standard limit they have encountered in the lecture. They eventually solve the version for x approaching zero by expanding the expression by x and substituting $x^2 = t$, that is, by producing esoteric domain text. However, they do not explicitly refer to the "multiplication rule" for limits from the lecture to justify their conclusion. They are not successful in their attempt to solve the second part of the task. As they adhere to a strategy to formalize their informal approaches and employ some methods suggested in the lecture, this can be seen as a production of descriptive domain text, which in parts, switches into the esoteric domain when they are trying out different algebraic transformations. Anne several times refers to "writing" it down properly, which indicates that she recognizes what type of text they are usually supposed to produce. The episode, in which the pair tries to solve the second part of the question, ends with a remark about the criteria for producing legitimate text:

Anne: Now you have made a writing mistake. You have to write X, or T, T goes towards infinity…They will like that at the mathematics department … also when we are very detailed.

The second pair also takes the interviewer's question as a prompt to produce expressive domain text by describing the meaning in terms of the weakly institutionalised oral discourse. Bob refers to the limit as "the value A when X is infinitely large, is a very very large number, don't know if I need to add more" and talks about "the little arrow" (see the transcript from the first account). After another prompt of the interviewer, they expand their explanation:

Ben: Yeah, the function value A as X tends to infinity, or? [silence ...] Then we have drawn [moving his hand as if he is drawing], have we not? [glancing at Bob]

Bob: Yes, it gets like that, x tends to infinity, it is very simple if you make a sketch [raising his hand with the pencil but does not draw, making drawing gestures while talking]. If we have A at a certain part of the y-axis we can say, we get such a horizontal line. The function starts at zero maybe and then goes up, kind of approaching A all the time, getting thinner, the bigger the x-value the closer you get ... and ... I don't know if I should bring that in too, you can always get closer than you already are, that is this thing with limits. That is the whole point, as in this case it will finally be as close as … you can't say as close as you can because you can always get closer but ...

They solve the task $\lim_{x \to 0} h(x)$ by reducing it to a standard limit, talking about substituting x^2 and decide about a solution. However, Ben seems unsure about the status of the solution produced by Bob (who does not refer to "multiplication rule"):

Bob: And this her goes towards zero, that X goes towards zero. One times zero.

Ben: One times a very small number next to zero.

Bob: This is what I also would like to say, indeed one times zero becomes zero.

Ben: I think we are clear with this one.

The last remark indicates that they do not adhere to the criteria for legitimate text established in the lecture. In the course of the solution of the second task, they remain in oral discourse and use visuo-spatial and movement metaphors for describing the shapes of standard functions and the "limit" as "approaching and coming closer". However, they are not successful in re-contextualising this discourse from the perspective the formal algebraic discourse. However, as the other pair, they seem to know the criteria for legitimate text, as Bob says at one occasion: "You can't do it like this mathematically /…/ It can be done, there is a method".

None of the pairs interpreted the first question of the interviewer as an invitation to establish the meaning for a novice by introducing her into the technical language and its institutionalized meanings, that is, to come up with a definition. Both pairs seem to realize that the legitimate text for successful participation in the course is located in the esoteric domain.

DISCUSSION

One goal of this exercise has been to see whether both interpretations can in combination produce useful insights about the students' reasoning about limits in the context of a university calculus course.

The first interpretation pictures those students showing an external sense of authority as the ones who tend to use the mathematical notations as keys to apply algebraic procedures. A conclusion could be that they lack an "intuitive feeling" for the mathematical objects involved, which should form the basis for using algebraic techniques. The second pair is pictured as showing an internal sense of authority and a preference for an "intuitive" approach. They often "know" by informal reasoning what the limit is and occasionally express a need to use algebraic representations. A conclusion could be that they lack an ability to use algebraic representations to formalise their reasoning. As the first approach focuses on the individuals' cognition it does not include the relation of their preferences to the context, in which these arose, as a specific research question.

The second interpretation shows that both pairs were, for different reasons, not able to produce solutions that would satisfy the criteria for legitimate text established in the lectures. The first pair did not have full access to the technical language and its institutionalized meanings, which they tried to employ, the second did not recontextualise their own productions from a formal algebraic perspective. This account draws attention to the structural complexities that relate to the ways in which the recontextualisation by means of formal algebra of the oral discourse about functions and limits employed in the lecture operates. It includes the establishment of a link of the students' productions to the discourse, in which they participate, as a *paradigmatic research question* (Radford, 2008) by conceptualising it in terms of their possession of recognition and realisation rules for producing legitimate text.

The two approaches also differ in terms of the methodology. While within the first framework the interview situation is a method for gaining insights into the students' beliefs and preferences, the second interpretation takes into account that the conversation during the problem solving sessions can also be conceived as a situation, in which the students are faced with the challenge of producing legitimate text. However, the students can neither have recognition nor reproduction rules for such a situation because it is the first time they participate in a study like this. They seem to have interpreted the interview situation differently, as more (Anne and Adam) or less (Bob and Ben) identical with the context of the course they were attending and thus more or less identifying the researcher with the official side of the university course. This interpretation would account for the fact that the second pair did not spend so much effort to translate their versions into a formal algebra as the first one and that they were mostly convinced that their solutions are reasonable, perhaps because of recognizing the context as informal. The first pair, in contrast to their following productions, engaged in weakly institutionalized discourse only as a response to the introductory question, perhaps recognizing the story about the imaginary friend as not belonging to the esoteric domain. In contrast, the first interpretation takes the students' explanations that follow the introductory question as an indication of their understanding of the concept of limit, or alternatively as an indicator of whether they know a definition in formal algebraic terms.

From the second perspective, "understanding" can be framed as having access to both of the discourses identified, as well as to the principle by which the oral discourse can be recontextualised from the perspective of the written one. The "intuitive" approach is only represented in the oral discourse. Both interpretations suggest a tension between these discourses that cannot easily be resolved.

It remains a highly questionable undertaking to look for combined insights stemming from interpretations that use languages of description, which stem from different theoretical traditions, particularly if issues of validity are at stake (cf. Gellert, 2008). The two interpretations presented here illustrate their points by selecting different episodes from the transcript. Considering that the research situation is re-interpreted in the second account (and thus taking the interviewer's questions as a piece of data), one could say that the two accounts are not interpretations of the same "data". In addition, different background information about the course has been used.

The outcomes of this interpretational exercise do not result in conflicting readings of the data. However, the results cannot be integrated into a combined insight, but only be juxtaposed. This is because the basic principles of the theories from which the approaches originate have established two different "universes of discourse" (Radford, 2008) in which the paradigmatic research questions are formulated.

REFERENCES

Alcock, L., & Simpson, A. (2004). Convergence of sequences and series: Interactions between visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, 58, 77-100.

Alcock, L., & Simpson, A. (2005). Convergence of sequences and series 2: Interactions between non-visual reasoning and the learner's beliefs about their own role. *Educational Studies in Mathematics*, 58, 77-100.

Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1011–1049). Charlotte, NC: Information Age Publishing.

Bergsten, C. (2007). Investigating quality of undergraduate mathematics lectures. *Mathematics Education Research Journal*, 19(3), 48-72.

Bernstein, B. (1996). *Pedagogy, symbolic control and identity: Theory, research, critique*. London: Taylor & Francis.

Dowling, P. (2007). Organising the social. *Philosophy of Mathematics Education Journal*, 21, online without pagination.

Gellert, U. (2008). Validity and relevance: comparing and combining two sociological perspectives on mathematics classrooms practice. *ZDM – The International Journal on Mathematics Education*, 40(2), 215-244.

Harel, G., & Trgalova, J. (1996). Higher mathematics education. In A. Bishop et al. (Eds.), *International handbook of mathematics education* (pp. 675-700). Dordrecht: Kluwer.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: RoutledgeFalmer.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM –Internat. Journal on Mathematics Education*, 40(2), 165-178.

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM –Intern. Journal on Mathematics Education*, 40(2), 317-327.

Silver, E. & Herbst, P. (2007). Theory in mathematics education scholarship. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 39-67). Charlotte, NC: Information Age Publishing.

COORDINATING MULTIMODAL SOCIAL SEMIOTICS AND AN INSTITUTIONAL PERSPECTIVE IN STUDYING ASSESSMENT ACTIONS IN MATHEMATICS CLASSROOMS

Lisa Björklund Boistrup, Staffan Selander

Stockholm University, Sweden

What can a multimodal social semiotic perspective in coordination with an institutional perspective make visible? In this paper we describe how we coordinate these two perspectives in order to look at the same empirical material with different focuses. The research interest is assessment actions in mathematics classrooms, an interest that also affects research objectives and possible results. When coordinating the different perspectives, we have chosen, for the analytical frame-work, to develop the social semiotic meta-functions by adding a new, fourth, meta-function: the institutional. For the detailed analysis, we connect to these four meta-functions other compatible concepts to create an analytical framework.

BACKGROUND

The focus of this paper is to describe how we coordinate two theoretical perspectives, multimodal social semiotics and an institutional perspective, in order to create a structured and nurturing analytical framework for the analysis of assessments during lessons in mathematics. We will start out by describing some of our central notions of assessment.

Assessment – a broad concept

Both in cases where some people realise that they actually are “capable” in mathematics, and in other cases where people think that they will never come to terms with it, we can notice “hidden” stories about assessment. Obviously, assessment explicitly takes place when students are given their mathematics test results. But often enough, assessment is implicit during teacher-student interaction in learning sequences. One example is the following: a student asks the teacher about a certain mathematical “rule” and wonders where it comes from. The teacher’s answer, by way of different communicational modes, shows that this particular student does not have to bother about such a question. S/he is just asked to follow the rule. But when another student asks the same question, the teacher engages in a discussion about the historical development of this particular rule. The first student in this example learns, through this implicit assessment, that the teacher does not consider her/him capable enough to understand this kind of question. Our assumption is that both the explicit assessments and the implicit assessments in mathematics classrooms play a key role for students’ learning. The empirical examples we use in this paper focus on implicit assessment actions.

COORDINATING TWO THEORETICAL PERSPECTIVES

As stated above, we hold that we are coordinating two different theories. Prediger et.al. (2008) make a distinction between *coordinating* and *combining* theories. They define "coordinating" as a term for bringing theories together that contain assumptions that are compatible, whereas "combining" is when the theories are only juxtaposed.

A multimodal social semiotic perspective

In a multimodal approach, all modes of communication are recognised (Kress et.al. 2001). Communication in a multimodal perspective is not understood in the same way as communication in a narrow linguistic perspective, focussing on verbal interaction only. Rather, all kinds of modes have to be taken into consideration, such as gestures, and gazes, pictorial elements and moving images, sound and the like. Relevant modes in (most) mathematics education are, for example, speech, writing, gestures and gazes as well as graphs, diagrams, physical objects, symbols, pictures and virtual animations. Modes are socially and culturally designed in different processes of meaning-making, so their meaning changes over time. It is also the case that one "content" in one kind of configuration (for example as speech), will not necessarily be the "same" content in another configuration (for example as illustration). Different representations of the world are not the "same" in terms of content. Rather, different aspects are foregrounded. In verbal texts we read linearly, within a time frame, whilst a drawing will be read within a space frame. And a graph does not represent a knowledge domain in the same way as numbers does. The modes that are "chosen" in a specific situation reflect the interest of the sign maker, and they are therefore not arbitrary. We argue for the importance of understanding multimodal communication to be able to fully understand a phenomenon as assessment. Language, in a broad sense, "may serve as a crucial window for researchers on to the process of teaching, learning and doing mathematics" (Morgan 2006, p 219).

We also argue that the assessment of learning (in a deeper sense) is about understanding *signs of learning*, as shown by different communicative modes (see Kress 2009, Pettersson 2007, Selander 2008b). This perspective is based on an understanding of learning as an increased engagement in the world, and as an increased capacity to use signs, modes and artefacts for meaningful communication and actions (Selander 2008a).

Institutional perspective

Within social semiotics, there are acknowledgements of institutional aspects, even though they are not always as clearly outlined as in the following:

> Detailed studies of the use of a given semiotic resource are interesting in their own right, but they also demonstrate a theoretical point. They show how the semiotic potential of framing is inflected on the basis of the interests and needs of a historical period, a given

> type of social institution, or a specific kind of participant in a social institution (van Leeuwen 2005, p 23, see also Morgan 2006)

Institutions are often taken for granted by the researcher who “knows” the situation. But without some idea of the communicative situation, it is very difficult to draw conclusions from, for example, a conversation. Here, we will go one step further in addressing “the institution” in its historical context. We understand that the interactions between teacher and student are situated in a context characterized by dominant mathematics education discourses, the use of artefacts developed over time, framings in terms of specific resources for learning, division of labour and time, established routines, classroom structure and authority.

Douglas (1986) argues that institutions (rituals, norms and classifications, what counts as centre or periphery etc.) affect the decisions made by individuals, for example the way they classify “phenomena” and “things”. Existing classification systems are often taken for granted. In this paper, we take the stance that classifications are products of social and cultural negotiations (Bowker & Star 1999). Wertsch and Toma (1995) emphasise that powerful institutional parameters constrain classroom discourse (see also Bartolini Bussi 1998, Lerman 1996). Our understanding of thee term institution is also to be seen as being in line with a dynamic view:

> Importantly, however, the thinking and meaning-making of individuals is not simply set within a social context but actually arises through social involvement in exchanging meanings (Morgan 2006, p 221).

Institutional framings have both direct and indirect effects. Decisions may be made on different “levels” in the school system, which have a direct impact on the classroom work. However, in this paper we will try to outline the indirect aspects, such as classificatory systems, norms and traditions developed over time. We will also use the institutional aspect already in the creation of analytical categories, not only as an overall umbrella-tool for reflecting over the results (see Björklund Boistrup 2007).

AN INSTITUTIONAL PERSPECTIVE IN RELATION TO META-FUNCTIONS

Inspired by Halliday (2004), social semioticians usually talk about three communicative meta-functions: the ideational, the inter-personal and the textual. In Morgan (2006), these functions are used with a focus on the construction of the nature of school mathematics activity. In this paper, we start out with the meta-functions as used by Kress et.al. (2001), focussing on assessment in mathematics.

As we see it, the three meta-functions are strong concepts for discussing situated communication and learning. However, two different kinds of restraints need to be noted. The first concerns the fact that not *all* possible communicative aspects can be captured by the three concepts. For example, expressive modes are not well captured (van Leeuwen 2005). Secondly, to be able to fully address institutional discourses in the situated communication and learning (as in this study), a wider notion of institutional framing (norms, institutional practices, classifications of good or bad performance etc.) seems to be needed. Communication in a classroom has different characteristics than communication in court or in a medical consultation. We add a fourth, *institutional* meta-function (proposed by Selander 2008c).

META-FUNCTIONS AND RESEARCH OBJECTIVES

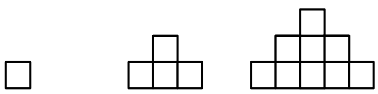
In this paragraph, we describe the four meta-functions and relate them to the research objectives of an ongoing research project on assessment actions in mathematics classrooms in grade 4 (10-year-olds). Even if all four meta-functions are present in all cases, in each and everyone of them, one function is in the foreground and the others are in the background. Thus, the division into four meta-functions related to four research objectives is meant to be seen as an *analytical* framework.

The ideational meta-function – aspects of mathematical competence

The *ideational* meta-function is related to human experience and representations of the world (Halliday 2004). When using this meta-function and aligning it with the *research interest* of assessment, the aim for the research project is to investigate what aspects of mathematical competence that are represented and communicated in the assessment actions.

In order to find a structure which can serve as part of the analytical framework for the more fine-grained analysis, we draw on a structure presented by Skovsmose (1990). He discusses mathematics education and the possibilities for mathematics to serve as a tool of democratisation in both school and society. He presents a structure of three aspects of mathematical competence:

- Mathematical knowledge itself
- Practical knowledge. Knowledge about how to use mathematical knowledge.
- Reflective knowledge. A meta-knowledge for discussing the nature of mathematical constructions, applications and evaluations.



First three figures in pattern.

In the following sequence, the students in the class are working in pairs on patterns. A boy (B) and a girl (G) are working together. Before the teacher approaches, these two students are discussing whether they need to count the squares one by one in order to find how many they are, or if they can use the pattern from an earlier task (1, 4, 9...). The excerpt shows what takes place when the teacher approaches the group. In the first line of the transcript, the students' speech (SS) and the teacher's speech (TS) are noted. In the next line, we find the students' and teacher's gestures (SG and TG), and in the bottom line the students' and teacher's body movements and gazes (SB and TB). The actions that occur simultaneously are written above each other. The teacher starts by asking how things are going.

SS: G: *25*	*Yes, it's going great!*

B: *This was strangely difficult.*

TS: *Are things going well?* *Why is it strange?*

SG: G is writing.

B stops writing.

TG:

SB: G is looking at her work and at T.

B looks at T and at his work. B looks at T.

TB: Approaches. Looks at G's paper. Moves close to G's desk. Looks at B's work. Moves closer to B's desk. Leans forward.

Time

We suggest that, during this lesson, the students get to show "Mathematical knowledge itself" related to patterns. The girl's comment that things are going great might be a sign that she feels that she has been able to handle the patterns well so far. The boy seems to have a different opinion. The teacher asks him and it becomes clear that this comment is mainly related to the aspect of mathematical competence focused on structuring one's notes. He has run into problems when drawing the figures:

SS: B: *You add this, but then it does not show that this one is this and that this one is this.*

TS: *No they are close now, but you can still see it I think. You'll have to leave more space between them.*

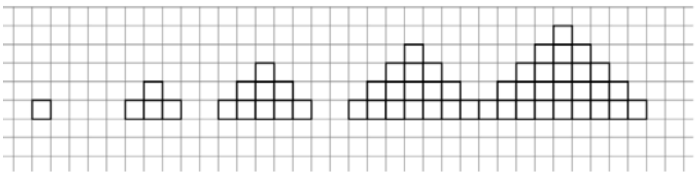
SG: B points at the figures on his paper.

TG:

SB: B looks at his work. B looks at T and down.

TB: Looks at B's work.

What he explains and shows by pointing is that two of his figures are drawn too close together on his paper, like this:



The teacher's comment is related to this "note-structuring" since she suggests that he should try to leave more space between the figures.

The interpersonal meta-function – feed-back, feed-up and feed-forward

The *interpersonal* meta-function is about how language (used in a broad sense in this paper) enacts "our personal and social relationships with the other people around us" (Halliday 2004, p 29). Morgan (2006) connects interpersonal aspects with assessment in an analysis of a classroom sequence. This is compatible with the way we use the interpersonal meta-function in this paper. Our *research interest* in relation to this is to find out what kind of assessment in the form of feedback and self-assessment is taking place in the interaction between teacher and student.

The structure for the detailed analysis is inspired by Hattie (2007). He suggests three kinds of feedback:

- feed-back – what aspects of competence has the student shown?
- feed-up – how can the aspects shown be related to stated goals?
- feed-forward – what aspects of competence might it be best to focus on in the future teaching and learning?

Using the same example as earlier, we find that the signs of assessment are shown both through the students' self-assessment and through the teacher's responses. Both the girl's and the boy's comments are within the category feed-back. The teacher's responses are connected both to feed-back and to feed-forward. We consider them as feed-back when the teacher communicates to the boy that his way of drawing the figures is acceptable; "No, they are close now, but you can still see it, I think". At the same time, she addresses a way of handling the very same issue during his continuing work, which we regard as feed-forward: "You will have to leave more space between them".

The textual metafunction – different communicative modes

The *textual* meta-function is related to the construction of a "text", and this refers to the formation of whole entities which are communicatively meaningful (Halliday 2004), in this case to other kinds of existing assessment systems and procedures. Teacher and students communicate in mathematics education with speech, gestures, gaze, pictures, symbols, writing and so on. According to this meta-function and our *research interest*, the objective is to investigate how different communicative modes (Kress et.al. 2001) are used and accepted by the teacher and the students. The boy shows his self-assessment on "note-structuring" by way of speech, gestures and drawings. The teacher listens and looks at the boy's work. Both the student and the teacher seem to accept different modes.

The institutional meta-function – tradition versus active participation

When it comes to institutional aspects of Swedish mathematics education, a dichotomous picture is often noticed (e.g. Palmer 2005, Persson 2006). On the one hand, the discourse of mathematics education is seen as "traditional", whereby students are expected to spend a good deal of time solely on solving all the problems in a textbook. On the other hand, the "wanted" discourse of mathematics education which emphasises a joint exploration in which, for example, students are invited to be active participants in problem-solving. These two discourses of assessment are similar to the discourses described in the literature on assessment in general (see Gipps 1994, Lindström & Lindberg 2005). The two discourses of assessment in mathematics can be summarised in the following way:

“Traditional” discourse	**“Active participant” discourse**
Focus on the correct answer	Focus also on processes
Focus on teacher’s guidance	Focus on the teacher promoting thinking
Focus on the number of finished tasks in the textbook in mathematics	Focus on the quality of the mathematical accomplishments
Focus only on the aspects of mathematical competence the student shows on her/his own	Focus also on the aspects of mathematical competence the student shows when working with peers
Focus only on written tests in mathematics	Focus also on documentation of the learning in mathematics
The teacher is the only one who assesses	The student is also part of the assessment

With inspiration from Lindström & Lindberg (2005).

In the following example, we keep to these dichotomous discourses. However, during the full analysis we will broaden the scope of discourses in relation to the findings. We will now go further on in the sequence from the classroom. We start out with the girl asking the teacher if it is possible for her to read what she has written and drawn on her paper. The teacher asks if the student understands it herself. The girl answers yes and the teacher says that she also understand the notes. Then the girl makes this comment:

SS: G: *Just so that you don’t mark it wrong “here you are wrong”*

TS: “laughs” *Is that what I usually do?*

SG: B & G are writing.

TG:

SB: B & G look at own work. G smiles.

TB: Looks at G’s work. Looks at B’s work.

As we see it, the girl’s comment refers to the traditional discourse of assessment in mathematics, since she proposes that the teacher might regard her notes as either wrong or right. The teacher engages in the discussion and asks if that is what the girl assumes that she as a teacher normally does. The girl answers no to this question and suggests that the teacher sometimes asks about notes that she does not understand. The teacher acknowledges this and the girl continues:

SS: G: *It is actually quite good to ask if you don’t know what the children have done* *Yes..*

TS: *Well, that is the only way to get to know.* *Mm*

SG: G & B are drawing. G stops drawing.

TG:

SB: G & B look at their work.	G looks at T	
TB: Looks at B's work.	Looks at G.	Nods.

Here, the other discourse is present, and by (finally) looking at each other, they seem to agree on this. To be able to assess the students' notes, the teacher might have to ask for clarification. The implicit assessment in this described activity is not just a matter of what is right or wrong. It is a matter of active participation by the student as well.

REFLECTIONS ON THE COORDINATION OF THEORETICAL FRAMEWORKS

We argue that the three meta-functions need to be understood in the light of institutional framings (also see Morgan 2006). The fourth meta-function is a way to both *understand* and *describe* institutional discourses as situated in history, and to address what it is that is at stake in conflicts and negotiations of assessment procedures and standards.

We find the theoretical perspectives described in this paper fruitful with regard to several aspects of the research process. We understand assessment as an act of meaning-making through a multimodal use of language. When defining the research objectives, the four meta-functions provide means to focus on different aspects of assessment actions.

In the short examples in this paper, we have shown how the aspects of mathematical competence that are present (the ideational meta-function) at first seem to be in patterns. But through the boy's speech, gestures and drawings, our understanding shifts to the structuring of notes. When it comes to the interpersonal meta-function. we find that both teacher and students show signs of feedback, and in the end the teacher also gives feed-forward. The textual meta-function gives us clues as to how the teacher and students use, and show acceptance of, different modes of assessments. Finally, the institutional meta-function makes it possible to describe the discourse as related to a strong tradition in mathematics education, but also in the ways in which new ideas can be ideationally, interpersonally and textually meaningful. In relation to this issue, we have described a situation in which the girl positions the teacher in a "traditional" discourse of assessment in mathematics (right-wrong). When analyzing what the teacher's gaze is focused on, we can notice that she initially looks at the boy's work when she is talking to the girl. But finally, when she turns towards the girl, they look at each other and the gazes reveal an "active participant" discourse.

This coordination of perspectives, including an analytical framework, seems to be a fruitful (and sufficient) basis for the full analysis of the empirical material in the project, in order to be able to describe, understand and discuss assessment in the mathematical classroom in a way that has not earlier been done (in Sweden).

REFERENCES

Bartolini Bussi, M. G. (1998). Joint activity in Mathematics classrooms: A Vygotskian analysis. In *The culture of the mathematics classroom*. Seeger, F., Voigt, J. & Waschescio, U. (Eds). Cambridge: Cambridge University Press.

Björklund Boistrup, L. (2007). Assessment in the mathematics classroom. Studies of interaction between teacher and pupil using a multimodal approach. In *Proceedings of CERME 5 22-26. February 2007.* (Retrieved from http://ermeweb.free.fr/CERME5b/, September 7 2008)

Bowker, G.C. & Star, S.L. (1999). *Sorting things out. Classification and its consequences.* London: The MIT Press.

Clarke, D., Keitel, C., & Shimizu, Y. (eds.). (2006). *Mathematics classrooms in twelve countries : The insiders perspective.* Rotterdam: Sense Publishers.

Douglas, M. (1986). *How Institutions Think.* Syracuse: Syracuse University Press.

Gipps, C. (1994). *Beyond Testing. Towards a theory of educational assessment*, London: The Falmer Press.

Halliday, M.A.K. (2004). Revised by Matthiessen, M.I.M. *An introduction to functional grammar*. New York: Oxford University Press Inc.

Hattie, J. (2007). *Developing Potentials for Learning: Evidence, assessment, and progress.* Key note speaker at EARLI 2007, Budapest, Hungary August 28 - September 1, 2007.

Kress, G., Jewitt, C., Ogborn, J. & Tsatsarelis, C. (2001). *Multimodal teaching and learning. The rhetorics of the science classroom.* London: Continuum.

Kress, G. (2009). "Assessment in the perspective of a social semiotic theory of multimodal teaching and learning. In Wyatt.Smith, C.M: & Cummings, J.J. (eds) *Educational Assessment in the 21st Century*. Springer.

Lerman, S. (1996). Intersubjectivity in Mathematics Learning: A Challenge to the Radical Constructivist Paradigm? In *Journal for Research in Mathematics Education*. 27(2), pp. 133-150.

Lindström, & Lindberg, V. (Eds). (2005). *Pedagogisk bedömning*. Stockholm: HLS Förlag.

Morgan, C. (2006). What does social semiotics have to offer mathematics education research? In *Educational Studies in Mathematics, 61(1), 219-245.*

Palmer, A. (2005) *Matematik i förändring. Diskursanalyser med. fokus på matematik och kunskapsteori med ett genusperspektiv*. Master's thesis. Stockholm: Stockholm Institute of Education.

Persson, E. (2006). *Det smyger sig in hela tiden. Om tankestilar och traditioner i utbildningen för blivande lärare.* Licentiate's thesis. Stockholm: Stockholm Institute of Education.

Pettersson, A. (2007). Pedagogisk bedömning – bedömning för lärande. I Olofsson, M. (red.) *Bedömning, flerspråkighet och lärande.* Symposium 2006. Nationellt centrum för sfi och svenska som andraspråk. Stockholm: HLS Förlag.

Prediger, S.; Bikner-Ahsbahs, A. & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education*, 40, 165-178.

Selander, S. (2008a). Designs for Learning – A Theoretical Perspective. In *Designs for Learning Volume 1 Number 1* March 2008. (Retrieved from http://www.designsforlearning.nu/current.htm, September 7 2008).

Selander, S. (2008b). Tecken för lärande – tecken på lärande. I Rostvall, A-L. & Selander, S. (Eds.) *Design för lärande*, 28-45. Stockholm: Norstedts Akademiska Förlag.

Selander, S. (2008c). Pedagogiska texter – ett sätt att designa lärprocesser. I Rostvall, A-L. & Selander, S. (Eds.) *Design för lärande*, 28-45. Stockholm: Norstedts Akademiska Förlag.

Skovsmose, O. (1990). Mathematical Education and Democracy. In *Educational Studies in Mathematics, 21: 109-128.*

van Leeuwen, T. (2005). *Introducing Social Semiotics*. London: Routledge.

Wertsch, J. V. and Toma, C. (1995). Discourse and learning in the classroom: A sociocultural approach. In *Culture*, *Communication and Cognition*. Wertsch, J. V. (Ed). Cambridge: Cambridge University Press.

INTEGRATING DIFFERENT PERSPECTIVES TO SEE THE FRONT AND THE BACK: THE CASE OF EXPLICITNESS

Uwe Gellert

Freie Universität Berlin, Germany

The paper contributes to the ongoing discussion on ways to connect theoretical perspectives. It draws explicitly on the introductory article and the concluding article of the Theory Working Group publication ZDM – The International Journal on Mathematics Education 40(2)*, particularly on the strategy of local theory integration. In the first part of the paper, a classroom scene is presented to provide some footing in empirical data. This data is used to illustrate the theoretical propositions, made from two theoretical perspectives, on the topos of explicitness in mathematics teaching and learning. In the second part, the two theoretical accounts are locally integrated resulting in a deepened and more balanced understanding of the role of explicitness. In the last part, this example is used to differentiate three modes of local theory integration: bricolage, recontextualisation and metaphorical structuring.*

PRELIMINARY REMARKS

According to Lakoff and Johnson (1980), the attribution of a front and a backside to something is metaphorical in nature and depending on the experience and interest of the attributor. A front-back orientation, they cogently argue, is not an inherent property of objects but a property that we project onto them relative to our cultural functioning. The front is what we see. If we want to see the back of it, we need to walk around it or to turn it round. This is quite clear for concrete objects like, say, mountains and fruits. Attributing a front-back orientation to the abstract concept of explicitness is different because there is no cultural agreement about what the front and the back of it may be. By projecting categories that emerge from direct physical experience onto non-physical constructs, a metaphorical structuring occurs which transmits the connotations of the former to the latter. It is thus no value-neutral endeavour to discuss the concept of explicitness in terms of its front and its back. In many cultures the front of something is regarded as being more important than its back, but otherwise the front may be taken as just a surface and you need to look at the back of it to see the 'real thing'. I will come back to some consequences of this issue, in terms of Radford's (2008) conceptions of theories, at the end of the paper.

In the paper, I present empirical data from a 5^{th} grade mathematics classroom for looking at the degree of explicitness in a case of mathematics teaching. I draw on the consequences of this teaching practice for the students' learning of mathematics from two theoretical perspectives, a semiotic ("the front") and a structuralist ("the back") one. While arguing that both perspectives connect fruitfully, I use this example for taking on the ongoing discussion of the challenges and possibilities of connecting theories in mathematics education (Prediger, Arzarello, Bosch & Lenfant, 2008).

THE EMPIRICAL DATA

In most federal states in Germany, primary school ends after 4th grade. From 5th grade on, the students are grouped according to achievement and assumed capacity. Those students, who achieved best in primary school, attend the *Gymnasium* (about 40% in urban settings). The data I am drawing on in this paper is the videotape of the first lesson of a new *Gymnasium* class, which consists of 5th graders from different primary schools. The teacher and the students do not know each other. It is the very first lesson after the summer holidays. The teacher starts the lesson by immediately introducing a strategic game known as “the race to 20” (Brousseau, 1975, p. 3). [1]

Teacher: Well, you are the infamous class 5b, I have heard a lot about you and, now, want to test you a little bit, that’s what I always do, whether you really can count till 20. [*Students’ laughter*.] Thus it is a basic condition to be able to count till 20, so I want to ask, who has the heart to count till 20? [*Students’ laughter*.] Okay, you are?

Nicole: Nicole.

Teacher: Nicole, okay. So you think you can count till 20. Then I would like to hear that.

>[2] Nicole: Okay, one two thr …
>Teacher: Two, oh sorry, I have forgotten to say that we alternate, okay?

Nicole: Okay.

Teacher: Yes? Do we start again?

Nicole: Yes. One.

Teacher: Two.

Nicole: Three.

Teacher: Five, oops, I’ve also forgotten another thing. [*Students’ laughter*.] You are allowed to skip one number. If you say three, then I can skip four and directly say five.

Nicole: Okay.

Teacher: Uhm, do we start again?

Nicole: Yeah, one.

Teacher: Two.

Both continue ‘counting’ according to the teacher’s rules. In the end, the teacher states “20” and says that Nicole was not able to count till 20. Then he asks if there were other students who really can count till 20. During the next 7 min. of the lesson, eight other students try and lose against the teacher whilst an atmosphere of students-against-the-teacher competition is developing. While ‘counting’ against the teacher, the tenth student (Hannes) draws on notes that he has written in a kind of notebook –

and he is winning against the teacher. After Hannes has stated "20", the following conversation emerges:

Teacher: Yeah, well done. [*Students applaud*.] Did you just write this up or did you bring it to the lesson? Did you know that today …

Hannes: I have observed the numbers you always take.

Teacher: Uhm. You have recorded it, yeah. Did you [*directing his voice to the class*] notice, or, what was his trick now?

Torsten: Yes, your trick.

Teacher: But what is exactly the trick?

During the next 5:30 minutes the teacher guides the mathematical analysis of the race to 20. In form of a teacher-student dialogue, he calls 17, 14, 11, 8, 5 and 2 the "most important numbers" and writes theses numbers on the blackboard. He makes no attempt of checking whether the students understand the strategy for winning the race. Instead, he introduces a variation of the race: you are allowed to skip one number and you are also allowed to skip two numbers. The students are asked to find the winning strategy by working in pairs. After 10 minutes, the teacher stops the activity and prompts for volunteers to 'count' against the teacher. The first six students lose, but the seventh student (Lena) succeeds. After Lena has stated "20", the following conversation emerges:

Teacher: Okay, good. [*Students applaud*.] Well, don't let us keep the others in suspense, Lena, please tell us how you've figured out what matters in this game?

Lena: Well, we've figured it out as a pair.

Teacher: Yes.

Lena: We have found out the four most important numbers and, in addition, the other must start if you want to win.

Teacher: Do you want to start from the behind?

Lena: From behind? No.

Teacher: No? Okay, then go on.

Lena: Okay, well if the other starts then he must say one, two or three. Then you can always say four. [*Teacher writes* 4 *on the blackboard*.] When the other says five, six or seven, then you can say eight. [*Teacher writes* 8 *on the blackboard*.] And when the other says nine, ten or eleven, then you can say twelve. [*Teacher writes* 12 *on the blackboard*.] And when the other says thirteen, fourteen or fifteen, then you can say sixteen. [*Teacher writes* 16 *on the blackboard*.] And then the other can say seventeen, eighteen or nineteen and then I can say twenty.

Teacher: Yeah, great. What I appreciate particularly is that you have not only told us the important numbers, but also have explained it perfectly and automatically. Yes, this is really great. Often, students just say the result, they haven't the heart, but you have explained it voluntarily. That's how I want you to answer.

In the next two paragraphs the focus is on the theoretical issue of explicitness. First, it is argued from a semiotic perspective that implicitness is a precondition for learning and that an exaggerated explicitness counteracts mathematical learning in school. Second, the structuralist argument that students benefit differently from invisible pedagogies is explored. The data is used to illustrate the theoretical propositions. [3]

THE FRONT: IMPLICITNESS AS A PRECONDITION OF LEARNING

From a theory of semiotic systems, Ernest (2006, 2008) explores the social uses and functions of mathematical texts in the context of schooling, where the term 'text' may refer to any written, spoken and multi-modally presented mathematical text. He defines a semiotic system in terms of three components (Ernest, 2008, p. 68):

1. A set o signs;
2. A set of rules for sign use and production;
3. An underlying meaning structure, incorporating a set of relationships between these signs.

According to this perspective, the learning of mathematics in school presupposes the induction of the students into a particular discursive practice, which involves the signs and rules of school mathematics. Whereas signs are commonly introduced explicitly, the rules for sign use and production are often brought in through worked examples and particular instances of rule application. The working of the tasks, the reception of corrective feedback, and the internalisation gradually enrich the students' personal meaning structures. It is only at the end when the underlying mathematical meaning structure is made explicit.

By referring to Ernest's semiotic system, we can make sense of the 5th grade teacher's actions: First, he is explicitly stating that counting the normal way till 20 is well-known for all students and he is playfully introducing a (growing) set of rules for sign use. Second, the strategies for winning the different races to 20 remain on an exemplary level and are not transformed into a general rule. Third, he leaves any exploration of the underlying meaning structure completely to the students.

Regarded from the adopted semiotic perspective, the teacher is inviting the students to a very open and not much routed search for regularities and more general relationships between signs. This way of teaching avoids what Ernest calls the "General-Specific paradox" (Ernest, 2008, p. 70):

> If a teacher presents a rule explicitly as a general statement, often what is learned is precisely this specific statement, such as a definition or descriptive sentence, rather than

> what it is meant to embody: the ability to apply the rule to a range of signs. Thus teaching the general leads to learning the specific, and in this form it does not lead to increased generality and functional power. Whereas if the rule is embodied in specific and exemplified terms, such as in a sequence of relatively concrete examples, the learner can construct and observe the pattern and incorporate it as a rule, possibly implicit, as part of their own appropriate meaning structure.

Apparently the teacher is introducing his mathematics class as a kind of heuristic problem solving. He is giving no hints for finding a route through the mathematical problem of the race to 20. When Hannes has succeeded in the race, the teacher is explicitly framing the solution as a “trick” that is useful in the particular task under study. He then continues by modifying the rules. This may allow the students to come closer to a general heuristic insight: It may be an appropriate strategy to work the solution back from 20. However, the teacher is not insisting upon Lena explaining backwards. The ‘official’ underlying (heuristic) meaning structure of the race to 20 is not made explicit during the lesson, though the students are gradually inducted into the generals of heuristic mathematical problem solving.

THE BACK: EXPLICITNESS AS A PRECONDITION OF ACCESS FOR ALL

From a structuralist position, Bernstein (1990, 1996) polarises two basic principles of pedagogic practice: visible and invisible. A pedagogic practice is called visible “when the hierarchical relations between teacher and pupils, the rules of organization (sequence, pace) and the criteria were explicit” (Bernstein, 1996, p. 112). In the case of implicit hierarchical and organisational rules and criteria, the practice is called invisible. He argues that in invisible pedagogic practice access to the vertical discourses, on which the development of subject knowledge concepts ultimately depends, is not given to all children. Instead, evaluation criteria remain covert thus producing learners at different levels of competence and achievement.

In terms of Bernstein’s differentiation of pedagogic practices, invisible practice dominates the 5th class’ first mathematics lesson. When comparing the teacher’s talk with Hannes and with Lena, it can be seen that the teacher keeps the students in the dark about some essential aspects of the mathematical teaching that is going on. Although students, who read between the lines of the teacher’s talk, may well identify some characteristics and criteria of the pedagogic practice they are participating in, the teacher transmits these characteristics and criteria only implicitly. All those students who do not notice these implicit hints, or cannot decode them, remain in uncertainty about:

… if the race to 20 is meant as a social activity of getting to know each other (It is the very first lesson!) or as a mathematical problem disguised as a students-teacher competition,

… if thus students should fish for “the trick” or heuristically develop a mathematical strategy and

… if thus successful participation in this classroom activity is granted when the race has been won or when a strategy has been established by mathematical substantiation.

Only at the end of Lena's explanation, the teacher makes the criteria for successful participation in 'his' mathematics class explicit. As a consequence, students' successful learning has been contingent on their abilities to guess the teacher's didactic intentions. Recording the numbers the teacher always takes (Hannes) without transcending the number pattern for a mathematical rule, is only legitimate to a certain extend. As long as the hierarchical and organisational rules and the criteria (which Bernstein (1996, p. 42) calls respectively the "distributive rules", the "recontextualizing rules" and the "evaluative rules") remain implicit, students are intentionally kept unconscious about the very practice they are participating in. Only visible pedagogic practices facilitate that students collectively access, and participate in, academically valued social practices and the discourses by which these practices are constituted (cf. Bourne, 2004; Gellert & Jablonka, in press).

CONNECTION: INTEGRATING THE TWO PERSPECTIVES

The contrasting perspectives on explicitness reveal that the rules and criteria of mathematics education practice remain – in part as a matter of principle – implicit. On the one hand, the need for implicitness is due to the very character of the learning process: whoever strives for whatever insight cannot say ex ante what this insight exactly will be. Ernest's "General-Specific paradox" is an interpretation of this issue. On the other hand, the principles that structure the practice of mathematics education remain implicit to the participants of this practice, without any imperative to do so for facilitating successful learning processes.

However, for that the general can be fully acquired, the students indeed need to understand that the specific examples and applications have to be interpreted as the teacher's means to organise the learning of the general. Successful learning in school requires the capacity to decode some of the implicit principles of the teacher's practice. The structuralist perspective supports the argument that the students actually benefit more from teaching-the-general-by-teaching-the-specific if they are conscious about the organising principle that is behind this teaching practice. By making the organisational and hierarchical rules and the criteria of the teaching and learning practice explicit, the teacher provides the basis for that all students can participate successfully in the learning process.

It is quite clear from the empirical data presented above that the teacher is partly aware of this relation: In the end of the passage, he explicitly explains to the students the characteristics of legitimate participation in 'his' classroom. However, as this explanation is given retrospectively and in a relatively late moment of the lesson it seems that some of the pitfalls of the implicit-explicit relation have not been avoided:

(1) It is neither obvious from their behaviour nor does the teacher check whether this very important statement is captured by all students. Particularly those students, who

did lose interest in the mathematical activity because they do not know where it can lead to, might not pay attention. (The fact that some students do not listen to the teacher's statement can be observed in the videotape.)

(2) By giving the explanation retrospectively, the teacher has already executed a hierarchical ordering of the students. Although no criteria for legitimate participation in the mathematical activity of the race to 20 has explicitly been given in advance of the activity, the teacher favours Lena's over Hannes' participation: Hannes is offering a "trick" (which might be more appropriate for playing outside school) while Lena is giving a mathematically substantiated explanation of her strategy. Apparently, Lena demonstrates more capacity of decoding the teacher's actions than Hannes does.

(3) It might be difficult for many students to transfer the teacher's statement to their mathematical behaviour during the next classroom activity. Indeed, the teacher is giving another specific statement, which the students gradually need to include in their meaning structure. This is another case of teaching-the-general-by-teaching-the-explicit: a general expectation ("students explain voluntarily") is transmitted by focussing on a specific example (Lena's explanation). Again, and on a different level, the students need to decode the teacher's teaching strategy: the teacher's statement is not only about legitimate participation in the race to 20, but also about participation in 'his' mathematics class in general.

Particularly the point (3) shows how the local integration of two theories may lead to a deepened and more balanced understanding of the issue of explicitness and its role within the teaching and learning of mathematics.

REFLECTIONS ON THE 'GENERAL': CONNECTING THEORIES

The connection of the two perspectives has structurally woven the front ("learning requires implicitness") into the back ("making hierarchical and organisational principles of classroom practice explicit"). A structuring of theoretical perspectives has thus taken place. But what is the nature of the new structure, and what are the characteristics of the process that has taken place?

Radford (2008) develops a conceptual language for talking about connectivity of theories in mathematics education. He takes theories as triples $\tau = (P, M, Q)$ of principles, methodologies and paradigmatic research questions. For questions about connectivity of theories, he argues that the principles seem to play a crucial role as "divergences between theories are accounted for not by their methodologies or research questions but by their principles" (Radford, 2008, p. 325). Indeed, at first glance, Ernest's semiotic perspective and Bernstein's structuralist perspective share an attention to the explicitness and implicitness of rules. The divergence of the two perspectives becomes apparent when the mode of these rules and their status is considered. Whereas from the semiotic perspective rules are rules for sign use and sign production and thus closely linked to the individual student's capacity of using and producing mathematical signs (P_1), the structuralist perspective takes rules as the constitutive elements of classroom practice (P_2). Ernest's semiotics is concerned with text-based activities where the texts are mathematical texts and the semiotic system is school knowledge. Bernstein's set of rules is the mechanism that provides an intrinsic grammar of pedagogic discourse. Although this looks like a fairly different understanding of rules and their respective theoretical status, the principles P_1 and P_2 of the two theories seem to be "'close enough' to each other" (Radford, 2008, p. 325) to allow for integrative connections.

Prediger, Bikner-Ahsbahs and Arzarello (2008, p. 173) describe "local integration" as one of the strategies for connecting theories. Acknowledging that the development of theories is often not symmetric, the strategy of local integration aims at an integrated theoretical account of a local theoretical question (e.g., Should rules be made explicit?). As a matter of fact, the principles P_i and P_j of two theories τ_i and τ_j deserve closer attention: How get P_i and P_j connected, what modes of mediating their divergence exist?

Bricolage. The mode of integration of theories Prediger et al. refer to is Cobb's notion of "theorizing as bricolage" (Cobb, 2007, p. 28). Cobb describes a process of adaptation of conceptual tools from the grand theories of cognitive psychology, sociocultural theory and distributed cognition. His goal is to "craft a tool that would enable us to make sense of what is happening in mathematics classrooms" (Cobb, 2007, p. 31). Here, the mode of mediation between theoretical principles is essentially pragmatic: Non-conflicting principles P_{g1}, P_{g2}, P_{g3}, … of the grand theories τ_{g1}, τ_{g2}, τ_{g3}, … are adapted for fit into the bricolage theory τ_b. As the goal of the integration is the development of a tool, τ_b is essentially an externally oriented language of description of empirical phenomena. Cobb's theorizing as bricolage is reminiscent of Prediger et al.'s (2008, p. 172) "coordinating" strategy. As the bricolage theory τ_b is a theory *en construction*, it is problematic to make the criteria for the selection of non-conflicting principles explicit.

Recontextualisation. Another mode of integration of theories is recontextualisation, "the subordination of the practices of one activity to the principles of another" (Dowling, in press, ch. 4). This is the case when the principles P_i of the theory τ_i dominate the principles P_j of the theory τ_j. An example of theory recontextualisation can be found in Gellert (2008) where an interactionist methodology M_i is subordinated to structuralist conceptual principles P_s. This process results in an asymmetrical role played by the methodologies M_i and M_s as a consequence of a hierarchical ordering of the principles of the corresponding theories (P_s over P_i; cf. Radford, 2008, p. 322f.). Hierarchical organisation of theories in the mode of recontextualisation is a device for avoiding theoretical inconsistencies.

Metaphorical structuring. A third mode of integration of theories is mutual metaphorical structuring. As Lakoff and Johnson (1980, p. 18f.) remark, "so-called purely intellectual concepts […] are often – perhaps always – based on metaphors". Since metaphors aim at "understanding and experiencing one kind of thing in terms of another" (Lakoff & Johnson, 1980, p. 5), this is again a case of subordination: metaphorical structuring. If we talk about the learning of mathematics in terms of rules, then the learning of mathematics is partially structured and understood in these terms, and other meanings of mathematics learning are suppressed. Similar things occur when concepts from one theory are infused into another theory. For an example see the infusion of the General-Specific paradox into the principles of a visible pedagogy. The argument that the advantage of a visible pedagogy relies on the explicitness of its criteria becomes differently structured when understood in terms of the General-Specific paradox: How can criteria be made explicit without producing blind rule-following and a formal meeting of expectations only? Infusing the term decoding capacity into the components of the semiotic system has produced a mutual effect: The teacher's strategy of teaching-the-general-by-teaching-the-specific is effective only if the students are able to decode the respective activities.

CONCLUSION

Bricolage, recontextualisation and mutual metaphorical structuring show different effects on the theoretical components that become locally integrated. This is still a complex issue and it might be very useful to further develop a meta-language for the connection of theoretical perspectives. I am convinced that a systematic description of the organising principles of local theory integration is an essential part of this developing language.

NOTES

1. The transcript presented, here, is my translation from the German original. Students' names are pseudonyms.
2. The sign > indicates overlapping of speech.
3. For a detailed analysis of what these passages can tell us about the exigencies that students face in mathematics classes, see Gellert and Hümmer (2008).

REFERENCES

Bernstein, B. (1990). *The structuring of pedagogic discourse*. London: Routledge.

Bernstein, B. (1996). *Pedagogy, symbolic control and identity*. London: Taylor & Francis.

Bourne, J. (2004). Framing talk: towards a 'radical visible pedagogy'. In J. Muller, B. Davis & A. Morais (Eds.), *Reading Bernstein, researching Bernstein* (pp. 61-74). London: RoutledgeFalmer.

Brousseau, G. (1975). *Etude de l'influence des conditions de validation sur l'apprentissage d'un algorithme*. Bordeaux: IREM.

Cobb, P. (2007). Putting philosophy to work: coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3-38). Greenwich: Information Age.

Dowling, P. (in press). *Sociology as method: departures from the forensics of culture, text and knowledge*. Taipei: Sense.

Ernest, P. (2006). A semiotic perspective of mathematical activity: the case of number. *Educational Studies in Mathematics, 61*(1/2), 67-101.

Ernest, P. (2008). Opening the mathematics text: what does it say? In E. de Freitas & K. Nolan (Eds.), *Opening the research text. Critical insights and in(ter)ventins into mathematics education* (pp. 65-80). New York: Springer.

Gellert, U. (2008). Validity and relevance: comparing and combining two sociological perspectives on mathematics classroom practice. *ZDM – The International Journal on Mathematics Education*, *40*(2), 215-224.

Gellert, U., & Hümmer, A.-M. (2008). Soziale Konstruktion von Leistung im Unterricht. *Zeitschrift für Erziehungswissenschaft, 11*(2), 288-311.

Gellert, U., & Jablonka, E. (in press). I am not talking about reality. Word problems and the intricacies of producing legitimate text. In L. Verschaffel, B. Greer, W. Van Dooren & S. Mukhopadhyay (Eds.), *Words and worlds: modelling verbal descriptions of situations*. Rotterdam: Sense.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago: The University of Chicago Press.

Prediger, S., Arzarello, F., Bosch, M., & Lenfant, A. (Eds.) (2008). Comparing, combining, coordinating – networking strategies for connecting theoretical approaches. *ZDM – The International Journal on Mathematics Education*, *40*(2).

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*(2), 165-178.

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM – The International Journal on Mathematics Education*, *40*(2), 317-327.

THE PRACTICE OF (UNIVERSITY) MATHEMATICS TEACHING: MEDIATIONAL INQUIRY IN A COMMUNITY OF PRACTICE OR AN ACTIVITY SYSTEM

Barbara Jaworski

Loughborough University UK and University of Agder, Norway

Theoretical perspectives of 'community of practice' and 'activity theory' are used along with constructs of 'inquiry' and 'critical alignment' to theorise developing mathematics teaching at university level. The paper introduces and explains the theories and relates theory to issues in the ongoing development of a mathematics course for engineering students. It focuses on developmental research which seeks both to chart developmental progress and lead to more informed teaching relating to the goal-directed activity of those involved, the systems of which they are a part and the tensions/issues within which development occurs.

INTRODUCTION

In recent writing (e.g. Jaworski, 2007, 2008a) I have focused on *communities of inquiry* in developing mathematics teaching and learning. I have drawn particularly on Wenger's (1998) concept of *identity* based in modes of *belonging* to a *community of practice*. This has been in the context of *developmental research* – that is research that seeks to develop practice while charting that development (see also, Goodchild, 2008). Here, I want to look more closely at how theoretical and methodological perspectives not only complement each other but are intertwined in the complex process of improving practice in teaching and learning mathematics.

I distinguish two areas of theory here. The first is Wenger's theory of *belonging to a community of practice*. The second is theory of *inquiry*, based in Vygotskian ideas of *activity*, *mediation* and *tools*. The complex notion of *identity* and its relation to *community* is a central unifying force. I have used these theoretical ideas previously to address analysis of data in a longitudinal study of developing mathematics teaching and learning in schools through collaboration between teachers and didacticians in Norway. Many sources document this research (e.g., Jaworski, 2007; 2008a; Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild and Grevholm, 2007; http://fag.hia.no/lcm/papers.htm). In this paper, I focus on the beginnings of research into developing mathematics teaching in a university mathematics department, focusing on my own practice as a (novice) mathematics teacher in this context.

The structure of this paper is as follows. First I give accounts, separately, of the two areas of theory, relating them explicitly to practices in mathematics teaching and learning. Then I turn to research into my own practice as a university mathematics teacher – a rather different form of practice from that of teaching mathematics in schools which has been my main focus in previous papers. I will expose some of the differences and related dilemmas and ways in which the two areas of theory cohere to support a theorising of practice and analysis of data. In doing this, I will address the nature of developmental research, its importance in contributing to development in mathematics teaching and learning, and issues in its operationalization

BELONGING TO A COMMUNITY OF PRACTICE

The term ‘community’ designates a group of people identifiable by who they are in terms of how they relate to each other, their common activities and ways of thinking, beliefs and values. Wenger (1998, p. 5) describes *community* as “a way of talking about the social configurations in which our enterprises are defined as worth pursuing and our participation is recognisable as competence”.

Within a university school of mathematics I recognize mathematicians, mathematics educators and our students at various levels as part of a community. In this community we engage with mathematics in various ways: learning mathematics, teaching mathematics and doing research into mathematics or into learning or teaching mathematics. Mathematics itself and what it means to do mathematics is central to this community. We can recognize both individuals and groups: that is to ascribe *identity* to both. Holland, Lachicotte, Skinner and Cain (1998, p. 5) write, “Identity is a concept that figuratively combines the intimate or personal world with the collective space of cultural forms and social relations”. Identity refers to *ways of being* (Holland, et al. 1998) and I talk here about ways of being in the university mathematical community. For example, people who teach mathematics have identity with relation to what it means to teach mathematics within a university environment, and within one particularly.

Within this community we all engage in some forms of *practice*: Wenger writes of practice: “The concept of practice connotes doing, but not just doing in and of itself. It is doing in a historical and social context that gives structure and meaning to what we do”. (1998, p.47). So *doing* within the school of mathematics means *engaging in the practice of university mathematics*. This *includes* doing mathematics, whether this is on the part of undergraduate learners or of research mathematicians; it *includes* students and academics researching aspects of the learning and teaching of mathematics, and associated contexts such as use of technology in teaching and learning and mathematics support for learners at all levels.

Wenger talks about identity in communities of practice as being about *belonging* to a community of practice. He suggests three modes of belonging: engagement, imagination and alignment. We *engage* in practice with others: our participation requires us to *do*, not just to observe the practices of which we are a part. Students have to engage with learning, teachers with teaching. All engage with mathematics. Engagement is the fundamental activity in doing. In order to engage we have to make sense of what we do; *imagination* allows us to interpret its various aspects and conceive of ways to achieve what we see as the goals of practice. We are not alone in our enterprise: the community of practice has developed over time and has norms and expectations of what will be done and how. We need to *align* with the norms of practice –

alignment provides the sociohistorical dimension within practice by which the practice is recognisable, sustainable and continuing.

Seeing university mathematics as a social practice is becoming a familiar basis for research in mathematics education related to learning and teaching mathematics in a university (e.g., Burton, 2004; Hemmi, 2006; Nardi, Jaworski & Hegedus, 2006) which has a long history and tradition, both in universities generally and in any one in particular. Recognisable aspects are university terms or semesters, lectures and tutorials, courses organised across several years of study in calculus, analysis, algebra and so on, and forms of assessment. Mathematics itself has an even longer history, with traditions in philosophical groundings, how topics are grouped and how learning and understanding mathematics are perceived. As mathematicians engage, whether in teaching or research, they bring imagination to interpret courses or research topics and they align with accepted practices, perpetuating a status quo and ensuring ongoing traditions. Students coming in fresh to the practices learn quickly acceptable forms of engagement and, imaginatively, how to make the system work for them according to their own, more familiar, communities of practice. They align with norms of practice developed over centuries and experience insights and obstacles familiar to cohorts of their forebears.

However, perpetuation of tradition is not always helpful in ensuring effective learning outcomes, especially if cohorts of learners no longer fit traditional moulds. Difficulties at the transition between school and university have been extensively reported (Hawkes & Savage, 2000). Existing research describes the mismatch between university lecturers' expectations of mathematics undergraduates and student competencies (London Mathematics Society, 1995). Brown, Wiliam, Barnard, Rodd & Macrae, (2002) reported how mathematics undergraduates' attitudes change and many become disillusioned with the style of teaching mathematics in university. In a study of teaching in university mathematics tutorials, Nardi, Jaworski and Hegedus (2005) suggested a variability of pedagogic awareness, in the teaching of university mathematicians, shifting from the naïve and dismissive to the confident and articulate. Hemmi (2006) studying mathematicians' and university students' attitudes to *proof* found distinct differences in the ways students and their teachers perceived mathematics learning and teaching at university level, and categorization of mathematicians interview responses showed significantly varying views on the nature of teaching. Burton's (2004) interview study of 70 mathematicians revealed both common traditions in mathematics teaching and research and particular viewpoints and idiosyncrasies. Such sources have highlighted both significant issues related to traditional practices and new concerns relating to changing traditions in which more research is urgently needed.

ACTIVITY, MEDIATION AND TOOLS: THE ROLE OF *INQUIRY*

Doing mathematics, for students at any level, requires engagement with abstract concepts which are not readily visible in the world around us. Although we can see particularities of mathematics in our familiar social worlds (examples of numbers or shapes, use of ideas of probability or statistical tools), expression of mathematical generality, necessarily, is abstract and requires abstract means of expression and justification. Schmittau (2003), drawing on Davidov, speaks of mathematics as involving *scientific concepts* which require "pedagogical mediation for their appropriation" (p. 226). Scientific concepts are concepts which cannot be learned spontaneously in engagement with everyday life (Vygotsky, 1986). Some form of mediation (going between) is needed for students to meet mathematical concepts and engage with them in meaningful ways. Particularly, Vygotsky talks about tools and signs which mediate the process of learning – *mediating artefacts* (see Figure 1). Such artefacts include both physical and intellectual tools; for example books and writing on paper, and language in which ideas and concepts are expressed. Technological tools can be helpful mediators for learning mathematics and teachers can orchestrate the use of technology to promote learning. *Pedagogical* mediation refers to the role of a teacher in creating opportunity for students to learn. The simple mediational triangle (Figure 1) deriving from Vygotsky and Leont'ev (e.g. Leont'ev, 1979) has been extended by Engeström (e.g., 1998)to include mediation *in social worlds* captured by the terms "rules", "community" and "division of labour" to which he refers jointly as "the hidden curriculum" (1998, p. 76). (See Figure 2). It is "hidden" because the factors involved are often not considered or questioned overtly as mediating factors in the education enterprise.

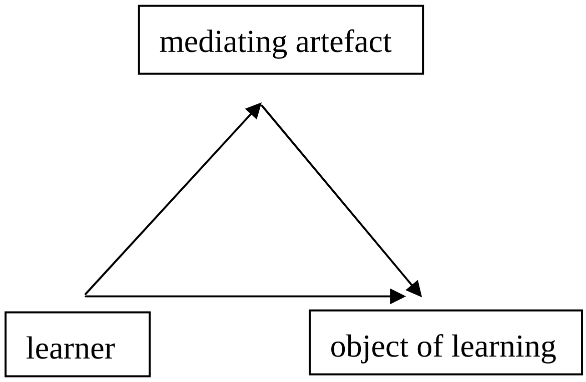


Figure 1: A simple mediational triangle

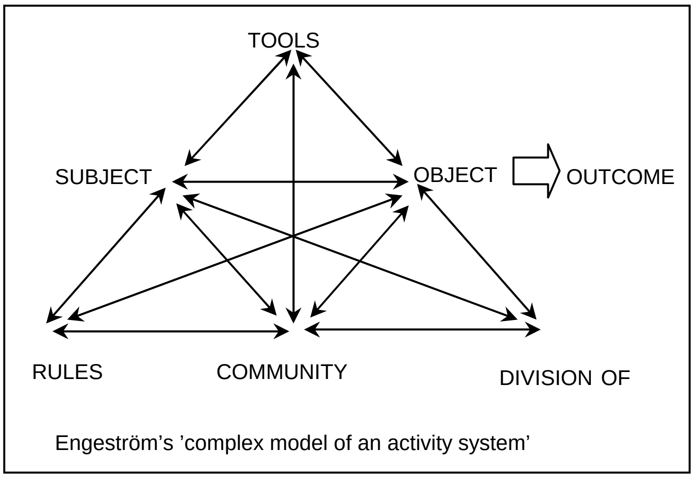


Figure 2: An expanded mediational triangle

In university mathematics education, the *rules* include courses to be taken, measures of success in a course or programme, expectations of participation; *community* encompasses those who engage in processes of mathematics learning and teaching with the purpose of advancing mathematical knowledge and understanding, primarily students and teachers; *division of labour* encompasses the differing roles and responsibilities of those within the community, for example teachers to teach and students to learn. Thus, for a learner (the *subject* of the learning process) with an *object* of learning mathematics, the activity of engaging in mathematics in a mathematical community is mediated by all of these factors as well as the artefacts commonly used to support learning.

Engeström refers to the system defined by the relationships illustrated in Figure 2, as an *activity system*, following a theory of *activity* deriving from Vygotsky and Leont'ev. Briefly, all activity is motivated, and comprises actions which are explicitly goal directed. Thus, in any such system, participants act according to goals and their actions are mediated by the various elements of the system (Leont'ev, 1979; Jaworski & Goodchild, 2006). An issue that arises in the learning and teaching of mathematics in a university is that of potentially conflicting communities where the goals of activity are concerned. So within a broad activity system of university mathematics (including students, teachers, researchers, learning, teaching and so on) we see subsystems which relate to the activity of certain groups. For example, teachers working within the established university system and its mathematical community have expectations of how students will act in relation to the norms and expectations of learning mathematics in a university. They have goals for students' learning and their actions are a consequence of their goals.

For students however, the system looks different. They come from different traditions in school systems and wider society. They are used to the kinds of relationships with teachers and peers that are afforded by pre-university education. They are highly influenced by popular culture and their peers. Stepping into the university system requires a re-alignment in their engagement; imagination, relating to the various communities of which they are a part, inspires their re-alignment. Lave and Wenger (1991) have offered a theory of legitimate peripheral participation to account for the transition for a novice into a community of practice. Here, I draw rather on Wenger's tri-partite characterisation of *belonging* and to activity theory to account for the dichotomies that emerge from collision of communities. Engeström's (1998) use of the expanded mediational triangle shows recognition of tensions in and between activity systems which can help address dichotomies. I say more on this below.

The place of *inquiry* in these theories and systems is central to my arguments in the paper. I see inquiry first of all as a *tool* mediating mathematics learning, teaching and development and then as *a way of being* in practice (Jaworski, 2006). When we start to inquire, we can be seen to use inquiry as a tool. Through sustained use the tool becomes a part of our identity as well, possibly, as of the identity of our community. Concepts relating to inquiry in practice, and its relation to these two established areas of theory, have emerged from 5 years of research in Norway (Jaworski et al., 2007). Seeing inquiry first as a tool emphasises its mediational characteristics within an activity system. Teachers and students, inquiring into the processes of learning and teaching, achieve “metaknowing” (Wells, 1999, p. 65ff) through inquiry practice. Inquiry in mathematics involves asking questions and working on problems which engage participants and lead to new awareness and ultimately knowledge – we see this both in the activity of research mathematicians (e.g., Burton, 2004) and, where an inquiry pedagogy is in place, in classroom mathematics. Inquiry in teaching mathematics involves teachers in asking questions and working on problems in didactics and pedagogy; inquiring into ways in which opportunity can be created fruitfully for mathematical learning. Inquiry is also central to a developmental research process in which research into aspects of learning and teaching mathematics leads to enhanced knowledge in the academy and, importantly, to more *informed* practice (Goodchild, 2008; Jaworski 2008a).

Seeing inquiry as *a way of being* shifts inquiry from its status as a tool, to a more fundamental constituent of an activity system in which it becomes part of the “hidden curriculum”, having a consequence of *making the hidden curriculum less hidden*. To manifest inquiry as a way of being requires inquiry to become part of the fabric of learning and teaching, what is taught and how it is approached, to such an extent that it permeates the rules, community and division of labour. It therefore offers a response to tensions and dichotomies that leads to metaknowing and possibilities for more knowledgeable practice. In order to explain this, I have introduced the concept, of *critical alignment*. Before discussing this in theory, I turn now to the context of university teaching and learning, and my own practice as a (novice) university teacher.

TEACHING MATHEMATICS TO FIRST YEAR ENGINEERING STUDENTS

At my university, the engineering faculty entrusts the mathematics teaching of its students to the *Mathematics Education Centre* which is the smaller of two parts of the *School of Mathematics*[1]. As I write this, I am currently in my second year of teaching a cohort of students in materials engineering some of whom have relatively low mathematical qualifications[2]. In the first year, I taught the weakest of these students (16 of them) separately from the rest and was able to develop good individual relationships. This year, all the students are together (around 70) and the approach to teaching is influenced strongly by this larger number. I want all students to be able to engage with mathematical concepts, to develop both conceptual understanding and procedural fluency and to be able to apply these to their engineering tasks. So, one area of inquiry is how I teach: what I do, how I do it, and what it achieves; included within this is encouraging students to inquire as part of their learning of mathematics. I bring an *inquiry way of being* as a result many years of experience, but nevertheless

[1] The other part is the Department of Mathematical Sciences. Members of both departments teach mathematics. Largely, those in the DMS do research in mathematics; those in the MEC do research in mathematics education.

[2] Some have not done mathematics beyond GCSE (the national examination at 16+). Others have very low grades in A level mathematics (the national examination at 18).

in this new arena I need to use inquiry overtly *as a tool*, both for myself and for my students. Methodologically, I engage in research and development cycles (Goodchild, 2008), planning, observing and analysing teaching and learning as it progresses; collecting data through teaching plans, reflective memos, student work, assessment tests, a student survey and student interviews.

Due to limitations of space here, I focus on just one aspect of teaching, for both year-groups of students. In the first year, to extend a more direct focus on curriculum topics, I offered a weekly investigative problem for students' exploration, requiring mathematical concepts with which students needed to develop strength and confidence[3]. It was introduced in a class session (we had two 50-minute sessions per week for 30 weeks); students were asked to continue to work on it in their own time, singly or in groups, and each one to give me some of their working and findings from the problem. Attendance at class sessions was very variable, but most of those who came handed in some work on which I wrote comments and returned to them. I learned about each student's mathematical skills and understanding from this activity. Observation over these weeks showed a willingness to engage with mathematics in non-routine ways on the part of more than half the students, and a classroom atmosphere in which questions could be asked and addressed and students mainly contributed actively (speaking up, asking questions, coming to the board) in class.

It became clear that some students had very weak mathematical skills, especially relating to algebra. When we came to the topic of exponential and logarithmic functions, I anticipated the difficulties that this topic would present. It seemed necessary to put all time and energy into the topic, and this halted the weekly problems. While maintaining an active questioning approach, I moved into a more direct approach to the topic: involving the class in sketching graphs, noting functional characteristics and relationships, expressing meanings aloud and addressing fundamental questions, and a strong emphasis on the rules of exponents and logs and their use in solving equations. Two outcomes were (a) in the related class test, several students achieved more highly than in two previous tests; (b) in a questionnaire in which I asked students to comment on their participation in the course, the level at which they rated their understanding of this material seemed more realistic and accurate than in relation to earlier topics. In my own reflections, while I was regretful of the demise of the weekly problem (it was not reinstated), I recognised that the teaching approach to *exp* and *log* had also achieved significant outcomes. I then had to rethink the objectives of my approach overall and their practical interpretation within constraints of time, curriculum and so on (Jaworski, 2008b). This has had implications for the current teaching. With a cohort of 70 the investigative problems with quick feedback would not be possible. The more direct approach has been maintained to a strong degree, and liai-

[3] For example, the *painted cube* problem which affords experience with algebraic formulation and manipulation – a wooden cube is painted on the outside and then sliced into smaller cubes all the same size; how many cubes have paint on one face, two faces, three faces?

son with the engineering department has started to produce problems relevant to the study of the particular students. An investigative element has been included using a *GeoGebra* medium.

The activity outlined above incorporated an *inquiry cycle* (plan → act and observe → reflect and analyse → feedback to planning) which led to growth and recognition of knowledge which should feed back into planning for teaching both locally and globally. Issues addressed included problems of variable attendance, a wide range of mathematical experience within the class, the time factor in focusing on a problem of the week, the demands of concepts that students found difficult and so on. *Aligning* within the university system was and is a necessity, but the element of inquiry has allowed a questioning of what is possible, experimentation and critical review of outcomes, and modification according to observation and analysis. This shows *critical alignment* in practice with related growth of knowledge and understanding.

An activity theory analysis shows some conflicts/tensions in these issues. For example, the problem of the week afforded development of confident mathematical participation and opportunity to work algebraically. The more direct addressing of mathematical concepts and associated skills afforded a greater achievement in curriculum-related summative assessment. Time and other factors militated against inclusion of both of these approaches. These issues can be seen as breaks in the mediating links in Engeström's triangle and highlight areas where the system is in conflict. Such conflict fosters the meta-knowledge that is needed to move forwards productively (e.g., Engeström 1998, p. 101; Jaworski & Goodchild, 2006).

I contrast here the two ways of theorising teaching development. Seeing *critical alignment in practice* emphasises *the inquiry process* in belonging to the community of practice which allows modification and change within engagement, imagination and alignment. The practitioner here brings an overtly critical eye to the practice and finds ways of adjusting her alignment. An activity theory analysis allows juxtapositioning of key elements of the activity system and examination of their relationships. Tools (e.g., the investigative problems), rules (e.g., lecture timetables), community norms (e.g., students who do not attend lectures) and division of labour (e.g., the expected roles of students and lecturers) can be seen to be in tension. Thus the analyst finds here a valuable tool in revealing the issues, their nature and relationship. This is both explanatory and predictive: it offers ways of seeing the status quo and reveals possibilities for consequent activity.

I see these two theoretical frames to have rather different functions. The first is closely related to action in practice: recognising where alignment is required and where it can be adjusted. It offers a practical interpretation in the use of inquiry as a tool, and aids development of an analytical awareness of how the inquiry cycle can both raise and address issues. The second allows a more holistic vision of the various factors and issues with a framework, a set of constructs, with which to characterise and link, and through which to see where the tensions lie. This allows further activity to be planned from the outside. Seen in these ways, the two frames offer complementary insights to the developmental process and the hidden curriculum.

THEORETICAL FRAMES AND ONGOING PRACTICE/ACTIVITY

One reviewer of this paper asked why students' goals had not been taken into account. This is an important question. With the first cohort of students, a questionnaire was completed asking about their course participation, understanding and achievement and some interviews were conducted (Jaworski, 2008b). Both cohorts completed the standard university evaluation of the course. In another research project into university teaching we have tried to organise focus groups with students to discern their perspectives. A discussion of analysis of these sources is beyond the scope of this paper. However, a future study would valuably bring students' goals to centre stage, particularly in an activity theory analysis in juxtaposition with teachers' goals. For example, in the use of GeoGebra as an exploratory tool, indications are that students do not so far see what the teacher perceives as value in its use. An activity theory analysis suggests that we have here tensions between the teacher's goals for creating conceptual understanding and students' goals for instrumental success. This could be shown by juxtapositioning of two activity systems, one for the students and one for the teacher. However, stronger data is needed before this would make sense. Critical inquiry into how GeoGebra can be used by students to achieve conceptual understanding is proposed as action.

REFERENCES

Brown, M. with Wiliam, D., Barnard, T., Rodd, M, & Macrae, S. (2002) *Student Experiences of Undergraduate Mathematics* (ESRC Report Ref. No. R000238564) obtainable upon request from the ESRC.

Burton, L. (2004). *Mathematicians as Enquirers: Learning about learning mathematics.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Engeström, Y. (1998). Reorganising the motivational sphere of classroom culture: An activity theory analysis of planning in a teacher team. In F. Seeger, J. Voigt & U. Wascgescio, *The culture of the mathematics classroom* pp. 76-103. Cambridge, UK: Cambridge University Press.

Goodchild, S. (2008). Reflecting the development of a Mathematics Teacher Educator and his discipline. In B. Jaworski & T. Wood (Eds.) *The mathematics teacher educator as a developing professional. Vol. 4 of The International Handbook of Mathematics Education.* Rotterdam: Sense Publishers.

Hawkes, T., & Savage, M.D, (2000). *Measuring the Mathematics Problem*. London: The Engineering Council.

Hemmi, K (2006). Approaching proof in a community of mathematical practice. Unpublished PhD thesis, University of Stockholm, Sweden.

Holland, D., Lachicotte, W. Jr., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Cambridge, Ma: Harvard University Press.

Jaworski B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education, 9*, 187-211.

Jaworski B. (2007). Theory in developmental research in mathematics teaching and learning: social practice theory and community of inquiry as analytical tools. In *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education*. Cyprus: Department of Education, University of Cyprus.

Jaworski, B. (2008a). Building and sustaining inquiry communities in mathematics teaching development: Teachers and didacticians in collaboration. In K. Krainer & T. Wood (Eds.) *Participants in Mathematics Teacher Education: Individuals, teams, communities and networks. Vol. 3 of The International Handbook of Mathematics Education.* Rotterdam: Sense Publishers.

Jaworski, B. (2008b). Helping engineers learn mathematics: a developmental research approach. *Teaching Mathematics and its Applications*, Vo. 27, no. 3.

Jaworski, B., Fuglestad, A. B., Bjuland, R., Breiteig, T., Goodchild, S., & Grevholm, B. (Eds.). (2007). *Learning communities in mathematics*. Bergen, Norway: Caspar.

Jaworski, B., & Goodchild, S. (2006). 'Inquiry community in an activity theory frame'. In J. Novotna, H. Moraova, M. Kratka, & N. Stelikova (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 353-360). Prague, Czech Republic: Charles University.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, UK: Cambridge University Press.

Leont'ev, A. N. (1979). The problem of activity in psychology. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology* (pp. 37-71). New York: M. E. Sharpe.

London Mathematical Society (1995). *Tackling the Mathematics Problem.* London: London Mathematical Society, Institute of Mathematics and its Applications, Royal Statistical Society.

Nardi, E., Jaworski, B., & Hegedus, S. (2005). A Spectrum of Pedagogical Awareness for Undergraduate Mathematics: From tricks to techniques. *Journal for Research in Mathematics Education* Vol 36 no. 4, pp 284-316.

Schmittau, J. (2003). Cultural-Historical Theory and Mathematics Education, in A. Kozulin, B. Gindis, V. S. Ageyev & S. M. Miller, (eds.), *Vygotsky's Educational Theory in Cultural Context*, pp.225-245. Cambridge: Cambridge University Press.

Vygotsky, L. S. (1978). *Mind in society*. Cambridge MA: Harvard University Press.

Vygotsky, L. S. (1986). *Thought and Language*. Edited, A. Kozulin. Cambridge, Ma: The MIT Press.

Wenger, E. (1998). *Communities of practice.* Cambridge: Cambridge University Press.

Wells, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education*. Cambridge: Cambridge University Press.

AN INTERPLAY OF THEORIES IN THE CONTEXT OF COMPUTER-BASED MATHEMATICS TEACHING: HOW IT WORKS AND WHY

Helga Jungwirth

Freelance scholar, Munich (Germany)

***Abstract**: I analyze the interplay of theories within a study on computer-based mathematics teaching. I will address divergences in their conceptualization of the empirical realities, influences on the interpretation of data, characteristics of my linking strategies, and issues of compatibility.*

Keywords: impact of theories on data analysis, theory development, compatibility of theories, micro-sociology, linguistic activity theory

INTRODUCTION

Amongst many others (Lester, 2005; Mason & Waywood, 1996; to name two only), "interpretative" research in the German speaking community of mathematics education has highlighted the crucial role of theory in research (Bikner-Ahsbahs, 2003; Jungwirth & Krummheuer, 2008; Maier & Beck, 2001). Accordingly, on the one hand, this research invests much in the development of theoretical frameworks, on the other hand, it aims at a development of locally limited, grounded theories. The outcome of research is thought of as a reconstruction of phenomena that is always theoretical in the sense that it transcends data and thus is an ideal type of reality (Bikner-Ahsbahs, 2003; Jungwirth, 2003). The Austrian research project "Gender – Computers – Maths&Science Teaching" by H. Jungwirth & H. Stadler was based on the above position. The aim was to reconstruct participants´ "relationships" to mathematics, physics and computers in computer-based classrooms, and the role gender plays within their interactive development (Jungwirth, 2008b; for the mathematics-related part). Apart from theorizing those relationships, a theoretical approach to classroom processes being appropriate for a comparison of both subjects had to be developed. It had to provide a notion of teaching as an ongoing process (in order to scaffold the investigation of the establishment of relationships) and as a whole (in order to be able to specify the contextual conditions of both subjects). My previous research suggested a use of micro-sociological theories and of a supplementary theory that was located in the context of activity theory. In this paper I want to deal with these theories and their networking restricted to mathematics teaching (Jungwirth, 2008a; for the related findings). As my aim is not to present the study itself I just mention briefly that the data consisted of 21 common Austrian, mostly CAS-based mathematics lessons, that all were videotaped and transcribed, and analyzed according to the standards of that "interpretative" research which means that interpretation follows hermeneutics and text theory in order to go beyond participants´ (i.e. teachers´) subjective understandings, and beyond everyday life readings of the analyzed events. The overall procedure to elaborate the final set of hypotheses is borrowed from grounded theory (Glaser & Strauss, 1967).

MICRO-SOCIOLOGICAL THEORIES

A micro-sociological perspective on mathematics teaching and learning has already proven fruitful in a variety of studies. To be precise, the attribute does not denote a single perspective but refers to different theories that share a basic understanding of social reality. Its structures are assumed to be established by the members´ of society mutually related acting. Those theories that figure in the project are symbolic interactionism (Blumer, 1969), and ethnomethodology (Garfinkel, 1967).

According to symbolic interactionism, interaction is the key concept to grasp social reality. Within interaction objects (anything that can be pointed, or referred to) get their meanings, and meanings are crucial for people´s acting towards objects and, in that, for establishing reality. Interaction is thought of as an emergent process evolving between the participants in the course of their interpretation-based, mutually related enactment. Thus, social roles, content issues, or participants´ motives as well are not seen as decisive factors; rather, they are also objects that undergo a development of their meaning. Consequently, neither the course of an interaction nor its outcome is predetermined. The term “interaction” is not restricted to events having outstanding qualities in respect to number of participants, topics, kinds of exchanges a.s.o. This means that classroom processes do not need to meet special demands in order to be a proper research object. From the perspective of symbolic interactionism, attention will always focus on the meanings objects get in local interaction, and on the very development of that interaction. As all participants matter from the standpoint of that theory, students are considered to be equally important as the teacher.

Ethnomethodology, too assumes that social reality is made into reality in the course of action but addresses the issue that despite of its formation social reality is taken as a given reality. This is due to the reflexive character of everyday activities. By accomplishing their affaires the members of society provide explanations for their doing and thus make it the normal way of doing. Ethnomethodology tries to reconstruct those methods. Accordingly, it helps in taking into account the methods by which teachers and students make computer-based mathematics teaching a matter of course whatever it will be about. Because of the shared stance towards reality the micro-sociological theories are treated here as “one” approach.

However, both theories are not sufficient. First, they address even large joint actions under the aspect of formation by separate acts of the participants; that is, they do not foreground the idea of a whole that has its specifics and thus can be spoken of as an entity. Hence it is difficult to think of teaching as a business that has an overall orientation. Secondly, both theories may induce a bias towards verbal events. There is a tendency to focus on verbal processes because of the prominent role of participants´

indications to each other which are indeed often verbal. Yet in an analysis of computer-based mathematics and, even more, experimental physics teaching all kinds of doing have to be covered.

LINGUISTIC ACTIVITY THEORY

The added theory (Fiehler, 1980) is a linguistic branch of activity theory (Leont´ev, 1978) that is not specialized on teaching and learning issues. Its basic concepts are activity, and activity complex. Activities are not merely actions but lines of conduct aimed at outcomes, or consequences. An activity complex can be thought of as a network of, not necessarily immediately, linked activities of some people that is oriented towards a material, or a mental outcome; that is, the concept always indicates a purposive stance. Linguistic activity theory in particular elaborates on the idea that there are three types of activities: practical activities (being accomplished by manipulations of material objects, or by bodily movements), mental activities, and communicative activities (in the sense of verbal activities). It foregrounds the interplay of these types of activities; actually between practical and verbal ones as the involvement of mental activities is a matter of inference. Two kinds of activity complexes – verbally, and practically dominated ones – are postulated in which the orientation towards verbal, or practical outcomes shapes the interplay in specific ways. As for my concern, linguistic activity theory helps me think of computer-based mathematics classrooms as entities having their own character. In particular, attention is turned to their global objectives. This is a relevant issue since in computer-based mathematics teaching IT plays an important role and could become a matter of teaching of its own right. Thus, there might be a further objective. The micro-sociological point of view is open to this option. But linguistic activity theory is in particular conducive to an identification of such cases as it helps in recognizing modes of activities and their interplay.

STRATEGIES FOR NETWORKING

As for the strategies of networking (Prediger, Bikner-Ahsbahs & Arzarello, 2008), “contrasting” theories has taken place so far and revealed that they play rather complementary roles. In particular, this holds for the micro-sociological approach on the one side, and for linguistic activity theory, on the other side. Each of them provides perspectives that are not covered by the other one but are needed to form a better whole: on situational adjustment and formation, on the one hand, and on certain aspects of structure and overall sense, on the other hand.

This two-sided approach has been used for a certain conceptualization of computer-based (mathematics) teaching: Its overall appearance depends in particular on predominating activities and objectives that are put into effect. These features give evidence of certain activity complexes that are the outcome of a multitude of similar negotiations among participants. Different types of computer-based mathematics teaching can be assumed to be established, ranging from a highly verbal teaching emphasizing mathematical aspects to a teaching that is totally devoted to carrying out manipulations at a computer. That conceptualization can be seen as a nucleus of a theory of computer-based mathematics teaching.

Thus, because of combining theories for the sake of the development of a local theory, synthesizing is a networking strategy in my research. The micro-sociological theories contribute by a “close-up”: the step-by-step formation of an activity complex becomes visible. Linguistic activity theory provides a “long shot”: a multitude of interactions can be spoken of and treated as an entity.

However, in order to elaborate that nucleus of a grounded theory it has to be applied to the data. Empirical phenomena are interpreted in its light. This means that the basic theories are also co-ordinated. Networking also serves the purpose to reconstruct concrete computer-based mathematics teaching. But as the research aims at a local, grounded theory, co-ordinating turns out to be synthesizing.

NETWORKING OF THEORIES: AN ILLUSTRATIVE EXAMPLE

The transcript is taken from an 11th grade classroom. During the lesson the class was given an introduction into maximum-minimum problems in which Derive should be used. The initiating task was: “A farmer has 20 metres of a fence to stake off a rectangular piece of land. Will the area depend on the shape of the rectangle?” A table should help to systematize the findings. In a first step, the students developed a conjecture based upon examples being subject of the first part (lines 01-26). In the following section of teaching (which is disregarded here) Derive was used to note the examples and to build the table. At the beginning of the second part (lines 134 ff) that table, containing columns for length (x), width (y), and area within the range of the examples, is visible to the students by a data-projector showing the solution of Erna who had to provide the official solution in Derive in interaction with the teacher.

01 Teacher:	Our question is. All these rectangles with circumference 20. Do
02 Sarah:	[inarticulate utterance]
03 Teacher:	they have the same area. For example which ones can we take.
04 Boy1:	No.
05 Boy2:	No.
06 Teacher:	Which range can you give an example length width
07 Boy:	Six and four?
08 Teacher:	Six times four is
09 Boy:	24
10 Teacher:	Another example
11 Eric:	Five times five this is the square
12 Teacher:	Five times five would be a square having which area
13 Eric:	25
14 Teacher:	Or a smaller one. Is there a smaller area as well
15 Carl:	For instance three times seven
16 Teacher:	Three times seven is 21. Or another one.

17 Carl:	One times two sorry one times ten
18 Teacher:	One times ten is ten or if we make it still smaller half a meter
19 Girl:	[inarticulate]
20 Teacher:	No. One times ten does not work one times nine would be OK. If the length will be ten what will happen.
21 Boy:	I see
22 Teacher:	Length ten what will we get if we take ten for the length
23 Arthur:	It is a line, a line [smiles], an elongated fence
24 Boy:	Not at all [continues inarticulately]
25 Teacher:	A double fence without an area thus the area can range from zero to. What was the largest so far
26 Eric:	25
<...>	
134 Teacher:	OK. This is OK. [to Erna] We can see if x is zero the width
135 Boy:	Ten
136 Teacher:	The area
137 Student:	Ten?
138 Teacher:	Yes. But now I like to have names for the columns x y z sorry x y the area. This we can do in the following way. We did it never before. Through a text object. Insert a text object [to Erna] this is not the proper place [it is above the table] but it does not matter no delete it. [she does] We want it below the table please click into the table and a text object above. Yes. And now you have to try. Use the cursor to place x y and area x in order that it is exactly above yes x y and the area. [she has finished] I do not know another way. I have figured out just this one. OK. We can see now the area change from zero 9 16 21 24 25 24. Hence the areas differ.

The episode 01-26 is about a response to a question. An analysis following symbolic interactionism can work out what participants´ taken-to-be-shared consensus concerning that response actually is. Participants deal with the question in the way that they first present a concluding answer (04, 05, maybe 02, too) and then demonstrate its correctness by giving several examples. Thus the response becomes a moot point again, and participants establish an everyday argument of the kind “statements about parts of a whole hold for the whole as well” (Ottmers, 1996) that confirms the initial response. As for the development of the interaction, specifying length, width, and area serves as a format for giving examples but the binding character of the format does not come about at once. For instance, the second student foregrounds his own point and brings into play the shape as well (11). The teacher is always just one party in an interaction. Also his dealing with the wrong combination of length one and width ten (20) is a reaction to the events.

Ethnomethodology enables me to reconstruct the ways in which the whole process of responding becomes a matter of course. For instance, students keep to presenting length and width as factors (11, 15, 17); or, in the case of disturbance (11), the teacher´s ineffective acknowledgement of the square, consisting of a confirmation and an immediate question about the area (12), proves appropriate for stabilizing the format. In the end, it is quite normal that responding is about making sure that the areas differ and about finding out their range. The reference to the square (although not irrelevant at all) turns out to be already beyond the established scope.

Both theories do not provide a more global understanding of the event. In particular, the question may arise what this episode is good for in the light of the research it belongs to. Linguistic activity theory helps to recognize a general purpose of the first part of the episode. It can be taken as a part of an activity complex: of an introduction to maximum-minimum problems. Accordingly, in the presented part a mathematical matter is made plausible that constitutes a problem that, in a generalized version, will have to be solved by means of calculus involving Derive. Besides, linguistic activity theory makes the solely verbal accomplishment of the response task a more remarkable fact; it springs to mind that, for instance, the table is not drawn on the blackboard. Conversely, however, this theory does not provide insight into the specific way of arguing that turns out to be the solution of this task in the end.

In a nutshell, in a co-ordinated theoretical perspective a mathematical event is established that has the role of a preparatory step in a computer-supported task solving. The subject matter-related potential of the interaction is realized as far as it answers this purpose of preparation though, in the light of that role, the pseudo-reasoning about the difference of the areas appears somewhat artificial. Participants produce that event through a fine, inconspicuous verbal adjustment of their acting.

At the beginning of the second part of the episode (134-137) participants demonstrate how the table has to be read. The values in the first line are used to explain what the output means. In a smooth-running process the teacher and two students establish a shared understanding of the table. After the reading has been clarified the table could be used (and this actually happens afterwards) to check the maximum area conjecture by further examples that are not confined to integer-sized rectangles (to be precise: an adapted version has to be used that provides numerical values in between). However, beforehand headings for the columns in the given table are produced. A second meaning of the table emerges. The table that was designed as a means for the solution of a mathematical task turns into a mere scheme being subject to completeness. The switch is initiated by the teacher, and shared by the students (for example, Erna´s immediate adjustment to the new task; 138). All the time manipulations are carried out, and the utterances refer to them. That makes a difference to the first part of the episode. There is much talking again but the accomplishment of the practical activities shapes the verbal process. The completion of the table in Derive becomes the subject of the episode. The situation offers an occasion for such a change; apart from that options of a program will always have to be introduced in some task context. However, as the table was already interpreted well and should help to systematize the findings, the switch is rather a surprise. But: If teaching in that introduction to maximum-minimum problems aimed at accurate products at a computer this turn towards the completion of the table would not be an extraordinary event. It just had to have priority then. This interpretation hypothesis grounding on linguistic activity theory would neither reject the possibility that those products at a computer could be conducive to mathematical ambitions nor exclude that there could be entirely mathematics-related negotiations. Thus, in its light the first episode need not be an exceptional event; it can even get an important role: it gives the computer-oriented business a mathematical air.

In a modified version, this hypothesis is the overall résumé of my research: Computer-based mathematics teaching of the observed type is a technologically shaped practice. The connection of the theories has also given insight into the particular features of that practice (Jungwirth, 2008a).

To combine theories of different grain sizes seems to be rather a successful strategy for co-ordinated data analysis and theory development (Prediger, Bikner-Ahsbahs & Arzarello, 2008; for some examples). In the following sections I want to address aspects of the theories featuring in my research that may further explain the fruitfulness of networking of theories in my case, and even beyond.

EMPIRICAL LOAD OF THEORIES

The first aspect is the “empirical load” of a theory (Kelle & Kluge, 1999). Accordingly, theories can be classed by the risk of empirical failure: whether or not they comprise concepts and statements from which categories and hypotheses can be deduced that can be examined, and thus refuted through data. In the first case a theory has empirical substance, in the second one a theory has no empirical substance. These are the poles of a spectrum of states.

Symbolic interactionism is at the second pole. It is a stance towards the world that can be hold, or rejected. It is not possible, for instance, to formulate refutable hypotheses for the position that objects get their meanings in the course of interaction, or to deduce categories for those meanings from the theory. Ethnomethodology too is a theory that lacks empirical substance, There is no empirical decision-making whether or not people´s methods to settle their everyday affairs make these commonplace affairs, and to fix in advance those methods.

Empirically empty theories have the role of “sensitizing concepts” (Blumer, 1954), that is, of mere perspectives from which data can be looked at. The outcome in the given case has to be worked out in the data analysis. Data can never make such a theory plausible; rather, conversely, interpretations of the data can be plausible in the light of the theory. Qualitative research often draws upon sensitizing concepts because they favour its approach to reality that tries to take into account participants´ own interpretations of that reality (Schwandt, 2000).

Linguistic activity theory has some empirical substance. Observable hypotheses can be built and examined through data. The category of activity and its properties “verbal” and “practical” can be used for this. For instance, it is possible to decide whether or not practical activities dominate and replace talking at all in certain manipulation contexts.

A use of empirically rich theories is characteristic, or even necessary, for quantitative research as the hypotheses to be formulated need a ground they can be deduced from. Within qualitative research referring to such theories may go beyond expectations concerning the rules for that kind of research. Accordingly, literature on methodology (Kelle & Kluge, 1999) points to the risk that properties of categories and hypotheses formulated in advance could dominate and interfere with the intended reconstruction of reality. However, it is not necessary to use empirically rich theories as it is done in quantitative research (Hempel, 1965); a researcher is not obliged to restrict her/himself to examinations of fixed properties and hypotheses.

My study gives evidence that empirically empty and empirically rich theories are compatible, and, moreover, that combining them is a practicable mixture. It seems that this does not hold in my case only. Such a constellation can make connecting theories on a level involving empirical analysis particularly effective. Certainly, applying solely theories without an empirical substance has proven fruitful in qualitative research (in mathematics didactics as well); however, it may be harder to elaborate typologies. Besides, empirically rich theories enhance the development of grounded theories as they help to carry out the check of interpretation hypotheses being strictly demanded in Strauss´ version of grounded theory (Strauss 1987).

CONCORDANCE OF BASIC ASSUMPTIONS (PARADIGMS)

The second notable aspect is the compatibility of basic assumptions theories make for the subject under investigation. To put this concern more clearly I present it in well-established terms: it is about theories´ belonging to paradigms. The concept of paradigm has quite a lot of meanings; I will adopt here the broad view of Ulich (1976) in which a paradigm is thought of as a socially established bundle of decisions concerning the basic understanding of the section of reality a theory wants to cover.

According to him, the duality of stability and changeability of social phenomena is a crucial aspect for theories that deal with social processes and settings. Consequently, he has made it a starting-point for a typology of paradigms. “Stability-oriented” paradigms regard regularities as manifestations of stable, underlying structures. Theories in that tradition try to grasp invariablities. “Transformation-oriented” paradigms ascribe regularities to conditions that are changeable because they are seen as having been established by the members of society. Thus, theories try to reconstruct the constitution of regularities and to find out conditions for change.

The theories I refer to differ in their origins and their concerns. Yet despite of all differences they share the idea that regularities are established regularities; that is, that they are outcomes of practice that can change if inner conditions change. This is obvious for the micro-sociological theories but it holds for linguistic activity theory as well. According to activity theory in general, society is a man-made society; order and stability of societal phenomena reflect the cultural-historical development of human labour and living conditions (although there is an inner logic in that development). Thus, all theories belong to the transformation-oriented paradigms. Symbolic interactionism and ethnomethodology are usually considered to be representative of the "interpretative" paradigm (Wilson, 1970) but that is, in the given typology, simply the micro-sociological version of the transformation-oriented ones.

This common ground justifies an approach to activity complexes under the aspect of local development and, as a consequence, the above conceptualization of computer-based mathematics teaching. If linguistic activity theory thought of human practice as an invariable, "given" entity, networking would not be honest at least. Actually, the idea that an interaction is determined by the roles of the participants, and the idea that an interaction is a negotiation process from which (also) roles emerge could not be combined to an integrated view on interaction serving as a base for analysis.

The general issue arising from the discussion above is which elements of their respective grounds theories have to share in order that networking on the level of some synthesis of theories, or of an integrated analysis, can take place.

To summarize: The last sections should shed some light on the compatibility of theories. It seems that it depends on, or at least benefits from the aspects addressed.

REFERENCES

Bikner-Ahsbahs, A. (2003). Empirisch begründete Idealtypenbildung. Ein methodisches Prinzip zur Theoriekonstruktion in der interpretativen mathematikdidaktischen Forschung. *ZDM*, *35*(5), 208-222.

Blumer, H. (1969) The Methodological Position of Symbolic Interactionism. In H. Blumer, *Symbolic Interactionism. Perspective and Method* (pp. 1-61). Englewood Cliffs: Prentice-Hall.

Blumer, H. (1954) What is wrong with Social Theory? *American Sociological Review*, *19*, 3-10.

Fiehler, R. (1980). *Kommunikation und Kooperation*. Berlin: Einhorn.

Garfinkel, H. (1967). *Studies in Ethnomethodology*. Englewood Cliffs: Prentice-Hall.

Glaser, B. G. & Strauss, A. L. (1967). The Discovery of Grounded Theory. Strategies for Qualitative Research. New York: Aldine.

Hempel, C. G. (1965). *Aspects of Scientific Explanation and Other Essays in the Philosophy of Science*. New York: The Free Press.

Jungwirth, H. (2008a). The Constitution of On-Computer Classrooms: Priority to Manipulation. http://www.helgajungwirth.homepage.t-online.de [Retrieved Aug, 10, 2008]

Jungwirth, H. (2008b). On the role of computers and complementary situations for gendering in mathematics classrooms. *ZDM Mathematics Education, 40*(4), 579-590, DOI: 10.1007/s11858-008-0116-x.

Jungwirth, H. (2003). Interpretative Forschung in der Mathematikdidaktik – ein Überblick für Irrgäste, Teilzieher und Standvögel. *ZDM, 35*(5), 189-200.

Jungwirth, H. & Krummheuer, G. (2008). Interpretative Forschung als Prozess: zu den Denkfiguren einer Forschungsrichtung von ihrem Beginn bis heute. In H. Jungwirth & G. Krummheuer (Eds.), *Der Blick nach innen: Aspekte der alltäglichen Lebenswelt Mathematikunterricht*. (Band 2, pp. 145-172). Münster: Waxmann.

Kelle, U. & Kluge, S. (1999). *Vom Einzelfall zum Typus*. Opladen: Leske + Budrich.

Leont´ev, A. N. (1978). *Activity, consciousness, and personality*. Englewood Cliffs: Prentice-Hall.

Lester, F. K. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *ZDM, 37*(6), 457-467.

Maier, H. & Beck, C. (2001). Zur Theoriebildung in der interpretativen mathematikdidaktischen Forschung. *Journal für Mathematikdidaktik*, *22*(1), 29-50.

Mason, J. & Waywood, A. (1996). The role of theory in mathematics education and research. In A. Bishop et al. (Eds.), *International Handbook of Mathematics Education* (pp. 1055-1089). Dordrecht: Kluwer.

Ottmers, C. (1996). *Rhetorik*. Stuttgart: J. B. Metzler.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM Mathematics Education*, *40*(2), 165-178.

Schwandt, T. (2000). Three Epistemological Stances for Qualitative Inquiry: Interpretivism, Hermeneutics, and Social Constructivism. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of Qualitative Research* (pp. 189-214). Thousand Oaks: Sage.

Strauss, A. L. (1987). Qualitative Analysis For Social Scientists. Cambridge, N.Y.: Cambridge University Press.

Ulich, D. (1976). *Pädagogische Interaktion. Theorien erzieherischen Handelns und sozialen Lernens*. Weinheim and Basel: Beltz.

Wilson, T. (1970). Conceptions of Interaction and Forms of Sociological Explanation. *American Sociological Review*, *35*, 697-710.

ON THE ADOPTION OF A MODEL TO INTERPRET TEACHERS' USE OF TECHNOLOGY IN MATHEMATICS LESSONS

Jean-baptiste Lagrange and John Monaghan

IUFM de Reims and University of Leeds

This paper examines why researchers adopt a theoretical model in reporting the results of their research. It describes the development of two researchers investigating teachers' use of digital technology in their lessons. The two researchers were dissatisfied in their attempts to understand the difficulties that the teachers they were researching experienced and they got round this dissatisfaction by augmenting their theoretical positions by the adoption of Saxe's four parameter model. The paper introduces Saxe's model, provides accounts of the researchers' development and ends with a discussion of issues raised.

INTRODUCTION

There has been considerable recent work on theories in mathematics education, reflecting researchers' efforts to be explicit about their theoretical assumptions and the links between different theories. CERME has been a focal point for many of these reflections. But why do researchers adopt a (particular) theoretical model in reporting the results of their research? There are many possible answers including: researchers are expected to adopt a theoretical model; a particular model may be 'in vogue'; the researchers work in a culture where a particular model is the accepted model; the model addresses central questions that the researchers seek to understand. We are two researchers, with different national backgrounds, who used Saxe's (1991) cultural framework and especially the four-parameter model to understand teachers' activities in using technology in their classrooms. We look at this model with regard to central issues we sought to understand. The paper addresses CERME Working Group 9's *call for papers* questions: *What divergences appear in the way different perspectives conceptualize empirical realities, tackle practitioners' problems? What is the influence of the different frameworks used on the research process? What is their influence on the interpretation of data?* The paper is a report of what Prediger (2008, p.285) calls 'problem solving "in the wild" of ordinary classroom practices' and considers the dual nature of this theoretical problem solving (theory and researcher). The paper first sets out Saxe's model, then describes why and how Saxe's model was used and ends by discussing issues arising.

SAXE'S MODEL

Saxe's model centres on emergent goals under the influence of four parameters: activity structures; social interactions; prior understandings; and conventions and artefacts (see Figure 1). Emergent goals are not necessarily conscious goals but are goals that arise from a problem in an activity and once the problem is solved the emergent goal usually vanishes. Saxe's model was conceived to explain mathematical practices in cultural transition (the Oksapmin tribe dealing with decimal money transactions) and is cultural-historical in its conception of artefact and interpersonal mediation in social practice. It has been applied in studies of street-sellers' practices (Saxe, 1991) and technicians' volume calculations (Magajna & Monaghan, 2003). It is, in our view, quite general in its application and particularly suited to the interpretation of innovative technology-based activity, such as teachers using digital technology due to unexpected goals emerging in this activity and the influence of cultural views regarding technology. The four parameter model is the first component of a three component theory: analysis of practice-linked goals; form-function shifts in cognitive development; the interplay of learning across contexts, i.e. Saxe's model is a construct and is part of Saxe's broader theoretical framework.

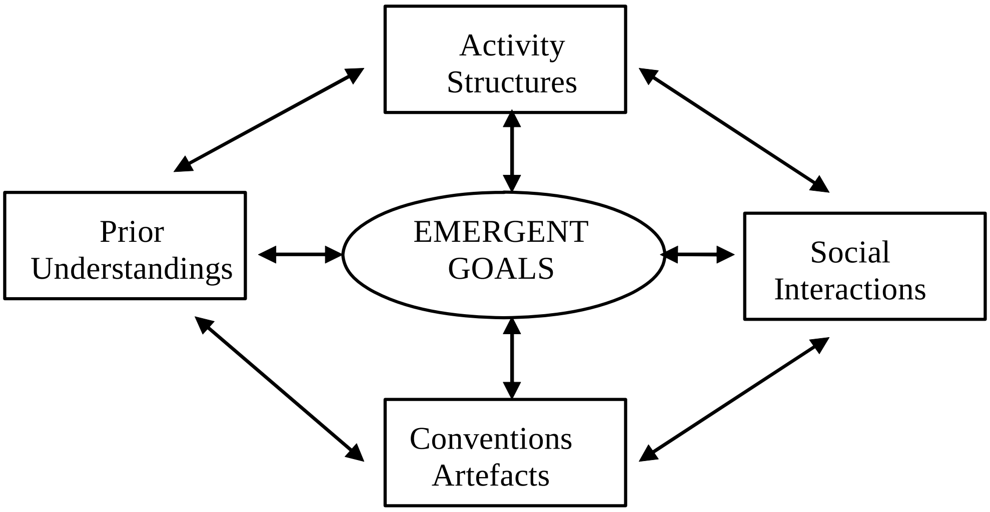


Figure 1 Saxe's four parameter model

We provide examples from Monaghan (2004) to illustrate the parameters, in the case of teachers using ICT, their interrelatedness and their impact on emergent goals.

The *activity structures* parameter "consists of the general tasks that must be accomplished in the practice- and task-linked motives" (Saxe 1991, p.17). In mathematics lessons this parameter concerns tasks that the teacher sets and the lesson structure. The tasks students engaged with in non-technology lessons were textbook exercises and the lesson structure was teacher exposition and examples followed by students doing textbook exercises. The tasks and cycles of the technology-based lessons varied considerably over the teachers and over time for each teacher.

The *social interactions* parameter concerns relationships between participants, teachers and students, in lessons and how these relationships influence participants' goals. It is very difficult to summarise differences between technology and non-technology lessons with regard to social interactions so we provide one example. Teachers spent much more time speaking to two or more students (as opposed to speaking to an individual) in technology lessons. Further to this the computer tools not only performed mathematical actions but also recorded the product of these actions and this provided a common basis for a group of students to collaborate.

The *conventions and artefacts* parameter, consists of “the cultural forms that have emerged over the course of social history” (ibid p.18). Cultural forms in mathematics lessons include techniques linked to traditional, not computer-based, tasks and tools and these can clash with new practices using new tools. A teacher using a spreadsheet planned a lesson focusing on ratio but the students’ and her emergent goals in the lesson were on getting the spreadsheet cells right, not only the correct equation but a suitable cell format. She commented after the lesson that she was unhappy with this focus on ‘cell-arithmetic’ and questioned “is this maths?”

The *prior understandings* parameter, includes teachers’ content, pedagogical and institutional knowledge, “the prior understandings that individuals bring to bear on cultural practices both constrain and enable the goals they construct in practices” (ibid p.18). The term ‘individuals’ is important because the different levels of experience participants in practice “bring to bear different (arithmetical) understandings on practice-linked problems and consequently their goals differ” (ibid., p.18). One teacher commented that with technology it was “back to being like a student teacher” because you are not prepared for any eventuality.

These parameters interact and impinge on practice-linked emergent goals. With regard to *conventions and artefacts* and *prior understandings* and the teacher who questioned whether cell arithmetic was mathematics, for example, this question was legitimate for her because her prior understanding of mathematics was formed in a public understanding of what (school) mathematics is. Further to this she voluntarily planned the task and wrote a worksheet which resulted in a focus on cell arithmetic and this discomfort only emerged in practice because her emergent goals in the lesson were shaped by the need to get the spreadsheet cells right.

HOW AND WHY WE CAME TO EMPLOY SAXE’S MODEL

We, in turn, state why we adopted Saxe’s model in our search for answers to central questions in our research.

Monaghan’s case

I have a long history of using digital technology in my own teaching and in working with other teachers who endeavoured to use it (some found it easy, others found it very difficult). In the late 1990s I ran a research project where I deliberately set out to work with teachers who had not used digital technology in their classrooms but who wished to do so. I worked closely with 13 secondary school teachers over a full school year, leading training sessions and conducting many interviews and observations. Teachers chose the technologies they would use which included computer algebra and dynamic geometry systems, graphic calculators and computer graphic packages and spreadsheets. Each teacher was video-recorded several times over the year (51 recordings in total) including one recording of a lesson at the beginning of the year where they did not use digital technology. Video-recordings were analysed using systematic classroom analysis notation (SCAN; Beeby et al., 1979). SCAN analysis involves viewing lessons as a series of activities, e.g. teacher exposition, students working, teacher-student dialogue. Each activity is viewed as a series of episodes, e.g. coaching, explaining. Events sub-divide the episodes into social and linguistic categories, e.g. managerial, confirmation. Coding consisted of categorising 30-second blocks with regard to the teacher, the students and the episode. I wrote and co-wrote a number of papers on this work but I still felt ‘unsatisfied’ – there were difficulties that the teachers had experienced in their practices that I could not explain in a satisfactory manner. In one paper (Monaghan, 2001), for example, based on SCAN analysis, I produced fairly strong empirical evidence that teachers using technology did not change from being ‘didacticians’ to ‘collaborators-with-students’ (as some constructivists would have it). I showed, for example, that many teachers became what I called ‘techno trouble shooters’ and I described the material basis for this (the set up and use of classrooms and computer-rooms) but this was not the deep understanding I was looking for.

Of the many intellectual influences on me at that time (≈2000), one that fitted with my thinking was Olson’s (1992) work on teachers’ routines. Olson views the study of teachers’ routines as a means to interpret teachers’ actions.

> Through classroom routines teachers express themselves. To understand what is being said in classrooms it is important to know what the routines are because such routines are rituals – performances involving significant symbols. These symbols belong to the tacit dimension of practice – what is said in the classroom that is not spoken directly.

As a teacher-educator who is familiar with teachers’ routines these words ring true to me but as a researcher in this project with teachers using digital technology I had a problem with a focus on routine – my project teachers, who were using digital technologies in the classroom for the first time, did not have routines – they were experimenting and doing lots of different things (according to the material conditions of their classrooms). I needed another means to interpret the difficulties my project teachers experienced and the diversity of in-class practices they exhibited. I had, with Zlatan Magajna, used Saxe’s model in his work on technicians’ mathematical practices and I considered analysing my project teachers’ practices via Saxe’s model. Initial considerations looked promising. I feel it is worthy to note, for discussion at CERME WG9, that this analysis via Saxe’s model was quite different to my SCAN analysis. The SCAN analysis was “local” in as much as it concerned categorising actions in specific (30 second) time intervals; further to this it was procedural and, as far as is possible in qualitative analysis, objective. The analysis via Saxe’s model was “holistic” in that whole lessons and often sequences of lessons informed categorisations and took the form of confirming or not the influence of parameters in teachers’ practices.

Lagrange’s case

My approach is to consider theories to address an overarching question: considering the potentialities of technology and the strong emphasis that society puts on its educational uses, why are these uses so rare, and why, when they exist, are they often deceiving? In this approach, I was brought to focus on the teacher using technology and especially on his(her) classroom activity, and to search for theoretical frames that could help in that endeavour. This approach is reflected in the contributions I wrote for CERME 2, 3 and 4 and in a recent paper (Lagrange, Ozdemir-Erdogan, to appear).

In CERME2 (Lagrange, 2002) I reflected on a meta-study conducted by a group of French researchers of a comprehensive corpus of international publications about research and innovation on the integration of technology into mathematics. The study built a framework of several dimensions in order to account for trends in the corpus. A statistical analysis provided evidence that dimensions considering the impact of technology upon the learner and mathematical knowledge were addressed by a wealth of studies and theories giving account of successes of the use of digital technologies mostly in ‘laboratory conditions’. The other dimensions related to the ‘ecology’ of technology in educational settings were poorly addressed in term of research studies as well as in terms of theoretical frameworks that could give account of successes but also of failures in ‘real school conditions’. We considered a ‘teacher dimension’ but found very few studies addressing this dimension.

In CERME3 (Lagrange, 2004) I focused on problematising teachers using technology. Returning to the overarching question of a discrepancy between the potentialities of technology and the actual uses, my interpretation was that innovators and researchers made an implicit assumption: new technologies and the associated didactical knowledge could easily be transferred to teachers by way of professional development and training. I thought that this assumption had to be questioned because, in a country like France, uses of technologies are deceptive although efforts have been made to train teachers. In my hypothesis the existing corpus of didactical knowledge and frameworks about digital technologies use was not sufficient to really help teachers integrate technology. Thus research had to study the teacher and try to look at his(her) action in the light of new frameworks.

Analysing research (especially Kendal &Stacey, 2001 and Monaghan, 2004) about the teacher and digital technologies strengthened the idea of a difficult integration, contrasting with research centred on epistemological or cognitive aspects. Kendal and Stacey brought evidence that, even in a research project, teachers’ use of technology can be very different to what was intended because of the influence of teachers’ beliefs and habits on the way they use technology in the classroom. Monaghan did a thorough analysis of teachers’ classroom activity showing that innovators’ expectations for a more open classroom management and for more emphasis on mathematics in teacher-students interactions were not fulfilled.

These studies were a first entry into the complexity of teachers’ relationship with technology use. To give account of this complexity and to think of new strategies for a better integration, I considered that an activity theory framework was needed. The reason is that, while teacher's activity in the classroom is problematic, it has its own logic and consistency. I believed that an activity theory framework would help to elucidate the difficulties encountered by teachers using technology in the classroom, while giving insight on how their activity and professional knowledge evolve during these uses.

In CERME4 (Lagrange, Dedeoglu & Erdogan, 2006) I tried out models of teachers' practices when using technology. Working with two doctoral students, observing and analysing teacher practices in two fields – teachers at lower secondary level using dynamic geometry and teachers at upper secondary level non-scientific stream using a spreadsheet, we (Lagrange, Dedeoglu & Erdogan) noted that classroom use of technology reinforces the complexity of teacher practices by introducing a number of new factors. Our aim was to understand the impact of these factors on systems of teachers' practices, and the conditions for classroom use of technology. We considered Robert and Rogalski's (2005) "dual approach" and we tried to complement this approach by using models dedicated to teacher use of technology: Ruthven and Hennessy's (2002) model addressed teachers' views of successful use, whereas Monaghan (2004) developed a model of teacher classroom activity inspired by Saxe (1991), as outlined above.

We noted in the conclusion that, combined with classroom observations, this model can help to make sense of phenomena in the classrooms that we observed. For instance, it is a general observation that teachers teaching in a computer room devote much time to technical scaffolding when they expected that technology would help their students to work alone and that they could act as a catalyst for mathematical thinking. Ruthven and Hennessy's model helped us to understand how a teacher can connect potentialities of a technology to her pedagogical needs, overlooking mathematically meaningful capabilities. The observation of two teachers using dynamic geometry showed what happens when the connection does not work: the teacher tries to re-establish the connection by becoming a technical assistant.

Saxe's model was chosen to appreciate teachers' specific positions using the parameters and to make sense of their classroom activity in similar lessons. We considered two teachers, one positively disposed towards classroom use of technology, and the other not, both of them experienced and in a context in which spreadsheet use was compulsory: a new curriculum in France for upper secondary non-scientific classes. We contrasted the two teachers through the viewpoint of Saxe's parameters and analysed their activity. In the classroom observations, we noted that teachers had to face repeatedly episodes marked by improvisation and uncertainty. The notion of emergent goals was central to analyse this flow of unexpected circumstances and questions challenging teachers' professional knowledge and parameters helped to understand how teachers react differently with regard to this flow. We also used other didactical constructs like *instrumented techniques* (Lagrange 2000) and *milieu* (Brousseau, 1997) that helped to highlight weak points in these teachers' activity: teachers seemed not to be able to open a clear dialogue with the students about why it is better to use spreadsheet techniques than usual paper pencil techniques. They also seemed to not have a clear view of the *milieu* they should establish for their teaching goals. Saxes' approach helped to understand the reasons for these weaknesses, mainly grounded in the different cultural representations between students and teachers (Lagrange & Erdogan to appear).

The analysis clearly separated the two teachers. One teacher was at an impasse. Her tendency to act on an exposition/application activity format and a teacher/student individual interaction scheme had been reinforced by the spreadsheet and consequently application was replaced by narrow spreadsheet tasks. With regard to individual parameters, the other teachers' dispositions towards technology integration were, in our opinion, excellent, but globally they conflicted and this teacher had to make real efforts to get herself out of such conflicts. Saxe's approach helped us to understand why good dispositions are not a guarantee of easy integration.

Using Saxe's model gave us more than what we expected. Because it is a cultural approach, it drew our attention to how cultural representations of the spreadsheet can differ, making it difficult for teachers to anticipate and understand what students do with the spreadsheet.

DISCUSSION

We consider issues raised above under two headings: the need for an augmented framework; how to evaluate the productivity of a theory.

The need for an augmented framework

Although we have developed as researchers in different countries we have, for many years, corresponded on matters concerned with the use of technology in the classroom. The constructs available to us, however, and in our opinions, for viewing teachers' activities in technology-based lessons were insufficient because they focused on teachers' established routines and technology messes up teachers' routines. Saxe's model, with its central emergent goals, provided us with a construct to view teachers' activities in technology-based lessons precisely because emergent goals arise from unexpected things that happen in such lessons.

A second reason for augmenting a theoretical framework lies in the gap between data analysis and data interpretation one can trust. Very often researchers conduct research with a framework that integrates methodology and theoretical approach, where data analysis leads the researcher to data interpretation. This appears very sensible unless one finds that the data analysis does not answer 'why' questions. This happened with Monaghan. SCAN analysis revealed large differences between teacher time spent (in technology and non-technology-based lessons) in teacher-whole class exposition, eliciting ideas from students, etc. (see Monaghan, 2001 for further details) but did not contribute to a deep understanding of why this was happening. Saxe's model, in Monaghan's opinion, provided a means to a deep understanding of these phenomena.

In augmenting a framework one should ensure that the augmentation is consistent with the underlying assumptions of the broader framework. In the case of Saxe and us there is a shared value of the importance of activity and mediation through artefacts and people. Further to this Saxe's model as a construct makes few assumptions. We have focused on emergent goals and parameters which interrelate with them. Emergent goals are ubiquitous in every human activity – so much so that we rarely notice them. Saxe's model has what Dawkins (2008), in discussing Darwin's theory, calls a large explanation ratio, 'what it explains, divided by what it needs to assume in order to do the explaining – is large'.

How to evaluate the productivity of a theory?

In our opinion two outcomes impinge on the usefulness of a theory or model, understanding and widening the research focus/questions. First, the theory or model should provide specific understanding with regard to the focus of the research. Comparing the contribution of Saxe's model to other frameworks helps to evaluate this specificity.

In Lagrange's national context two frameworks are dedicated to learning (Theory of Didactical Situations, Anthropological approach) and a framework is dedicated to the teacher (Robert and Rogalski's (2005) 'dual approach'). These frameworks were useful, but the conclusions we drew did not constitute sufficient progress towards understanding the situation of teachers using technology.

As said above, considering how teachers dealt with the "milieu" and the spreadsheet techniques helped to highlight weak points in their activity. But it was not our central question. The question was why it is specifically difficult, even for experienced teachers, to develop a consistent activity when using technology. Then, the question is, why are those teachers not aware of these weaknesses, or, if they are, why do they not change their activity? Saxe's framework provided a means for a deeper understanding of these weaknesses: rather than a poor didactical analysis, they reflect teachers' uncertainty, and differences between students and teachers, with regard to spreadsheet representations and the fact that it was difficult for teachers to anticipate or understand what students do with spreadsheets.

Robert and Rogalski's approach assisted a consideration of the complexity of teachers' activity. We learnt from that that we would have to consider a plurality of factors with complex links between them. We anticipated and observed that, rather than bringing solutions, technology amplifies complexity. This result is, however, too general and did not account for the uncertainty experienced by teachers using technology in the classroom. The 'dual approach' postulates that practices are complex *and* stable, that is to say that teachers' practices do not change easily because they are constructed to deal with the complexity. In contrast, teachers' practices in dealing with the complexity of classroom use of technology are far from stable and Saxe's framework assisted an analysis of this unstability as a flow of emergent goals.

A second criterion for the useful contribution of a theory or model is that it helps to widen the research questions. The main reason for choosing Saxe's model was the uncertainty of teachers' activity when using technology and the need for a holistic approach of this activity. We were attracted by the model rather than by the whole framework: goals and parameters seemed adequate to analyse teachers' classroom activity, and they actually were. But after using the model, we reflected why this model was productive. We realized that there should be something in common between our teachers and the New Guinea Oksapmin from which Saxe built the model. This should be that both had to deal with a new artefact involving deep cultural representations. In the Vygotskian perspective, Saxe was interested by the impact of culture upon cognition and he chose the Oksapmin people because in their case there was a conflict of cultures: these people have a traditional way of counting, using parts of the body as representation of numbers; some of them trade in the modern way, but their traditional way does not permit them the calculations that this trade requires. This comparison brought us to consider cultural systems involved in classroom use of technology. Students saw the spreadsheet as a means to neatly display data. It is consistent with the social representations of technological tools. People are generally not aware of the real power of the computer, which is the possibility of doing controlled automatic calculation on a data set, even when they used spreadsheet features based on this capability. In contrast, the teachers saw the spreadsheet as a mathematical tool. They were disconcerted because they were not conscious of the existence of other representations. Clearly, Saxe's approach helped us to widen our reflection about the impact of cultural views associated to computer artefacts upon classroom phenomena.

REFERENCES

Beeby, T., Burkhardt, H. & Fraser, R. (1979). *Systematic classroom analysis notation for mathematics lessons*. Nottingham: Shell Centre for Mathematical Education.

Brousseau, G. (1997). *Theory of Didactic Situations in Mathematics. Didactique des Mathematiques 1970-1990.* Dortrecht: Kluwer Academic Publishers.

Dawkins, R. (2008). Why Darwin matters. *The Guardian* (February). Retrieved September 30, 2008 from http://guardian.co.uk/science/darwinbicentenary.

Kendal, M. & Stacey, K. (2001). The impact of teacher privileging on learning differentiation with technology. *International Journal of Computers for Mathematical Learning*, **6**(2), 143-165.

Lagrange, J.-B. (2000). L'intégration d'instruments informatiques dans l'enseignement: une approche par les techniques, *Educational Studies in Mathematics* 43, 1–30.

Lagrange J.B. (2002). Looking at the potentialities of Computer Algebra Systems in the light of a meta-study on the use of IC Technologies to teach and learn mathematics. Proceedings of CERME 2, Marianske, Cz Republic.

Lagrange J.B. (2004). The mathematics teacher and the integration of ICT. Proceedings of CERME 3, Bellaria, Italy.

Lagrange J.B., Dedeoglu, N. & Erdogan E. (2006). Teachers using technology: models of the complexity of practices. Proceedings of CERME 4, San Feliu, Spain

Lagrange J.B. & Ozdemir-Erdogan E. (to appear). Teacher's emergent goals in spreadsheet-based lessons: analysing the complexity of technology integration. To appear in *Educational Studies in Mathematics*.

Magajna, Z. & Monaghan, J. (2003) Advanced mathematical thinking in a technological workplace. *Educational Studies in Mathematics*, **52**(2), 101-122.

Monaghan, J. (2001). Teachers' classroom interactions in ICT-based mathematics lessons. pp.383-390 in M. van den Heuvel-Panhuizen (ed.) *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3.

Monaghan, J. (2004). Teachers' activities in technology-based mathematics lessons. *International Journal of Computers for Mathematical Learning* 9(3), 327-357.

Olson, J. (1992). *Understanding teaching: Beyond expertise*. Milton Keynes: Open University Press.

Prediger, S. (2008). How are theoretical approaches expressed in research practices? A report on an experience in comparing theoretical approaches with respect to the construction of research problems. *ZDM – The International Journal on Mathematics Education*, 40, 277-286.

Robert, A. & Rogalski, J. (2005). A cross-analysis of the mathematics teacher's activity. An example in a French 10th-grade class, *Educational Studies in Mathematics*, 59, 269–298.

Ruthven, K. & Hennessy, S. (2002). A practitioner model of the use of computer-based tools and resources to support mathematics teaching and learning, *Educational Studies in Mathematics*, 49, 47–88.

Saxe, G. B. (1991). *Culture and cognitive development: Studies in mathematical understanding*. Hillsdale NJ: Laurence Erlbaum Associates.

THE JOINT ACTION THEORY IN DIDACTICS:

WHY DO WE NEED IT IN THE CASE OF TEACHING AND LEARNING MATHEMATICS?

Florence Ligozat & Maria-Luisa Schubauer-Leoni

FPSE, Université de Genève (CH)

In this paper, we reflect on the Anthropological Theory of Didactics and the Theory of Didactical Situations in Mathematics as the roots of an emergent framework: the Joint Action Theory in Didactics. Disclosing some of the boundaries of the two major French theories in didactics allows us to sketch an integrative scheme of certain of their principles and concepts within the background of socio-cultural and pragmatist approaches to teaching and learning practices.

This paper aims at contributing to the discussion that has progressively given rise to a "theory networking space" in the previous Working Group sessions. We regard this work as an important step for several reasons. First, it accounts for the paradigmatic partition of the main theories currently used in mathematics education, ranging from the more *cognitive* ones that focus on the understanding processes of individual learners, to the more *cultural* ones, that are oriented by institutional and collective structures in which knowledge is subjected to social transactions. It sheds a new light on certain theories we are familiar with, since they are contrasted with some others on certain aspects like the role of social interaction, the role of learning environments, the role of the teacher…etc. Second, some very interesting mechanisms are disclosed about the ways researchers may attempt to connect theses theories, while preserving their specificities. We especially value the tension between *integration* possibilities and *boundaries* to preserve, but also the triplet [principles, methodologies and paradigmatic research questions] that is worked out by Radford (2008).

As we support the development of comparative studies in didactics, these questions are of premium interest for delineating both the generic and the specific (i.e. content knowledge related) principles of the intricate processes of teaching and learning. More particularly, the work in progress in this CERME Working Group is an opportunity for us to reflect on the development of the Joint Action Theory for Didactics (JATD), for the purpose of grasping teaching and learning complexity under ordinary classroom conditions.

PART I : SKETCHING A NETWORKING SPACE FROM ATD AND TDSM

In the first part of this paper, we contrast the two major theories developed by the French didactics of mathematics, *i.e.* the Anthropological Theory of Didactics (ATD; Chevallard, 1985/1991; 1992) and the Theory of Didactical Situations for Mathematics (TDSM; Brousseau, 1997). Since these frameworks have developed over more than 30 years, this has to be drastically reduced to their major orientations, without having here the opportunity to decline the various branches that they inspired further on. Indeed, what we are most interested in is their epistemological stance rather than outlining these theories *per se*. In line with one of the most important principle underlying both the ATD and the TDSM, we consider that theories, like knowledge, emerge as a collective elaboration to face a set of problems and questions that human groups experience in the development of societies. Thus, a good starting point for inquiring into theories may be to compare the realm of reality they account for, through *their paradigmatic research questions* (Radford, 2008) along with their *epistemological roots* in human sciences.

From an historical standpoint, the theorization of an "experimental epistemology for mathematics" that was worked out by G. Brousseau in the mid 70's is a mean to account for the generation of meaningful mathematical knowledge in classrooms. Then, in the early 80's, Y. Chevallard's anthropological analysis of the conditions of knowledge dissemination within institutions, shed a new light on knowledge taught as reworked from its genuine context of emergence in expert (or academic) communities. Therefore, the knowledge coherence and legitimacy as presented in school, has to be studied in terms of epistemic affordances and constraints. In both cases, the epistemological account of the knowledge content at stake as the third pole of the didactical system opened the era of the didactics of mathematics as a science taking off from the psycho-pedagogical stance on teaching and learning.

Since the early works, the ATD relied upon an assumed structuralist point of view of knowledge development within institutions that can be referred to the background of a Durkheimian sociology and eventually to certain socio-cultural approaches. In line with Douglas (1986), the basics of the ATD are that (1) ways of thinking of individuals are shaped by the collective practices to which they partake and (2) these collective practices are oriented by purposes whose coherence defines the primary goal of an institution as a social organisation bound to achieve a type of task. In the case of educational institutions, the transmission of a socially agreed culture is the core of the activity, relayed by an "intention to teach" and an "intention to learn" at the level of the teacher and the students respectively. Thus, the determination level of what the participants do is to be studied in the institutional patterns of the teaching and learning culture. Early works from Chevallard (1985/1991) have stated that the way mathematical knowledge is ordinarily presented within educational institutions does not match the epistemological way the mathematics are built (*i.e.* the mathematical praxeologies in the ATD). Differences in goals generate differences in tasks to be achieved and so the patterns of school mathematics are somewhat distant from academic mathematics. The *transposition process* as the starting point of the ATD accounts for the specific organisation of knowledge in the purpose of its transmission within educational institutions. In particular, the didactical transposition process is characterised by (1) a *decontextualisation* of mathematical practices from the problems they originally attended, into sequence of topics to fit the curricula constraints and the frames of teaching time; (2) a *recontextualisation* of these topics by the teachers, in order to make the students encounter the knowledge to be taught within the classroom practices. This process has long been regarded to be consubstantial to the functioning of didactical systems as ruled by institutional practices. In recent works, it has been refined by featuring the didactical praxeologies as a set of practices, combining with each other, in order to describe the possibility of studying the process of mathematics in the classroom. It is structured in terms of moments that are *theoretically inherited* from the praxeological structure of the mathematical knowledge, *i.e*. the two levels of practices that correspond respectively to the techniques for solving a type of problem and the formulation / justification of these techniques. In furthering this, the ATD also attempts to account for the role of words, graphics and gestures as "ostensive objects" that shape the mathematical activity. Ostensives encapsulate the socio-cultural definition and values of the mathematical knowledge and they provide tools for a praxeology to develop. In our view, the ATD's paradigmatic research questions attend to a *top-down systemic approach* of the mathematical studying process. A description of the mathematical tasks and the possible didactical praxeologies are attempted as forms of institutional practices.

The epistemological roots and the research questions of the TDSM are more complex to depict. Brousseau's well-known starting point is that a given mathematical knowledge can be functionalised by *a fundamental situation* gathering the epistemological conditions for the emergence of the considered piece of knowledge in the human culture. This major underlying principle is somewhat compatible with the definition of the mathematical praxeologies in the present works of the ATD. Whereas ATD considers this principle as a mean to *describe the possible structures of human practices in studying mathematics*, at the level of institutions, the TSDM refers to the same principle for *modelling the epistemological conditions* in which the students may develop some meaningful mathematical knowledge, within the classroom.

A major concern in G. Brousseau's work is to identify such fundamental situations in the primary school mathematics and to derive some didactical situations from them. In such situations, students encounter some constraints requiring an adaptation of their prior knowledge towards the learning of a new one. The students have to work out the solution of a problem in which specific knowledge cannot be avoided. Brousseau explicitly refers to the Piagetian theory of learning. The core of the learning process relies upon the students' adaptation to *a milieu* as a set of epistemological constraints. The milieu is designed to orient the students' actions by providing some positive or negative feedbacks to the strategies used. To achieve meaningful learning, the students have to take the responsibilities of their game (devolution) without relying on the teacher's feedbacks. This is what Brousseau defines as *an a-didactical situation*, in which the student is supposed to focus his/her interest on a "game" against the milieu and "forget" the teacher's expectations at least for a while. From the student's point of view, the outcome of the game is a new "connaissance" that is being progressively socialised within the classroom debate. Typically, the student first *acts* to find a local solution to the problem, then *formulates* his/her strategies through a communication game and finally, the strategies may be *validated* within a controversial debate in the classroom. Moving from the peculiar answer to the problem to a generalised pattern of knowledge is supported by *some changes in the milieu* with which the student interacts. Then, the institutionalisation process managed by the teacher makes sure that the "connaissance" constructed by the students within the didactical situation, is adequate to the definition of knowledge in curricula. Thus, the outer horizon of Brousseau's didactical situations remains coherent with a cultural approach of knowledge. However, the kernel of this theory relies upon *a constructivist epistemology* where the student-milieu relationship primes the learning process, by the mean of the a-didactical situation. Social interactions come into play for anchoring the "connaissances" built by students as individuals, within the pre-existing sociocultural knowledge. As noticed by Radford (2008), they are "a mere facilitator of individual's development of mental structures"(p320). In our view, the paradigmatic research questions that the TDSM addresses is *the design of epistemic models of knowledge*, *i.e.* situations that enable an adaptive shift of the student towards the construction of new knowledge, without relying onto the teacher's indications at some points of the didactical contract.

Both these theories attempt a model of teaching and learning mathematics as a three poles system where the "being teaching" (teacher) and the "being taught" (student) are two epistemic instances constrained by the knowledge structure. In the ATD framework, the diffusion of mathematical knowledge is studied merely at the collective level of the social structures whereas the TDSM attempts to link the conventional patterns of knowledge and the connaissances constructed by individuals in a rather functionalist way (the milieu originates in the student's actions /formulations/validations). These structural and / or functional stances on the teaching and learning process were crucial in the development of the French didactics of mathematics. We regard it as a major epistemological break from the merely psychological approaches to students' difficulties in mathematics and the pedagogical positivism more generally. It afforded the premises of a science of the teaching and leaning phenomena in mathematics, and it also inspired other subject matter didactics in the French speaking community. However, moving back to the major features of each theory allows to highlighting some irreducible boundaries between them.

<u>The epistemological boundary:</u> The TDSM draws strongly on the student – milieu interactions, as an epistemic model of the adequate conditions for reconstruction of knowledge to occur within didactical conditions. The teacher's role in the devolution and the institutionalisation phases is an add-on. In between, the teacher organises the constraints of the milieu to sustain the optimal interactions. The dualistic relationships between the student and the milieu exclude the vision of the classroom social environment as a "thought collective" (Douglas, 1986) to which each student is subjected *ipso facto* through the use of language and more generally signs that are socially agreed. *The predominance of the milieu, as a pre-structured environment made of material, symbolic and social objects to which students have to adapt themselves, shadows the reflective activity that they may also activate to make meanings from*

collective practices. The adaptive function of the milieu addresses the individual minds as independent structures that become intertwined through the formulation and validation games. The reference to the collective practices is not continuous in the participants' experience as it is supposed to be in the underlying principles of the ATD framework. However, one can also argue that the ATD focuses on the institutional practices mainly but the way individuals may get the ownership of these practices and eventually make them evolve, is not accounted for. Very few elements describe what the participants effectively do within the didactical system, in order to teach and learn. As stated by Arzarello, Bosch, Gascon & Sabena, "the non-ostensive objects exists because of the manipulation of the non-ostensive ones within specific praxeological organizations" (2008, p181). *The interpretative process of the collective meanings by individuals are shadowed by the schemes of institutional practices that (over)structures local purposes and psychological processes*. Although the concept of "mesogenèse" was promisingly introduced (Chevallard, 1992) to account for the dynamics of the relations between individuals and objects in their environment, it did not deepen, for instance, how the semiotic systems handled by students (i.e. ostensives) may generate meanings, i.e. non-ostensives (Schubauer-Leoni & Leutenegger, 2005).

<u>The methodological boundary:</u> Early works from Chevallard stated that, ordinarily, the knowledge presented to students in classrooms does not appear according to the epistemological conditions in which it was born, due the decontextualisation and sequentialisation processes in curricula. From this point of view, the works carried out by Brousseau's team may be regarded as an attempt to *counter* the transposition process by redesigning school mathematics into meaningful situations that are not ordinarily supported by didactical institutions. Indeed, a didactical situation is supposed to restore some of the epistemological conditions for knowledge to be built, by designing specific learning environments. A series of fascinating designs were produced in which cultural knowledge is genuinely functionalised (numbering with integers, measuring capacities, introducing rational and decimal numbers, Euclidean divisions, linear functions…etc.). But the way ordinary school institutions may incorporate these situations is not investigated, leaving some opportunities to misleading interpretations of certain examples of didactical situations in some teaching materials. Furthermore, the design process tends to minimize the teacher's work which is then strongly supported by the research team. One can say that it shunts the *"repersonnalisation" process* of the institutional patterns of knowledge, which is ordinarily carried out by the teachers. The relationships between the milieu to be organised and the interaction arena which is ruled by the reciprocal expectations of the didactical contract is the main concern. But the relationships between the ordinary resources that the teachers use and the effective teaching environments they implement cannot be investigated from Brousseau's paradigmatic research questions because they strongly rely upon research designs.

From these boundaries, we argue that (1) the TDSM cannot be regarded as a direct continuation of the ATD framework in terms of classroom practices and interactions among individuals ; (2) the structuro-functionalist stances that are consubstantial to both these theories does not allow an account of the interpretative motions of the subjects within the didactical system as an social institution. These two points could be said to be out of synch with the purposes of those researchers who actually work with one or another theory. Nevertheless, we argue that if didactics is to be a science of the teaching and learning phenomena about a given content knowledge, then some new research questions have to be addressed.

PART 2 : THE GROWTH OF J.A.T.D. AS AN INTEGRATIVE THEORY

In this part, our purpose is not to feature details and examples of use of the Joint Action Theory in Didactics, since this is presented in Sensevy (this group of papers). We rather would like to present the conditions of emergence of its paradigmatic research questions and how some principles and concepts may be borrowed from the ATD and TDSM, by the mean of a conversion process in the light of some pragmatist theories to match a socio-historical perspective of knowledge development in teaching and learning (Forget & Schubauer-Leoni 2008; Ligozat, 2008).

Many empirical studies have reported that the specific role played by the milieu in TDSM's is a feature that is hardly observed as controlled by the teacher in ordinary classes. Most of time, the set of objects partaking to the situation is not self-sufficient to enable students develop an epistemic relation to the problem or task to be achieved. Or, to reformulate this in the terms of the ATD, consistent bodies of mathematical praxelogies are hardly managed by the teacher. However, in these ordinary conditions, that we consider to be the most common teaching and learning reality for mathematics, we cannot envision that no learning happens at all. It progressively leads us to consider that *didactical situations that would be a priori endowed with some a-didactic affordances may not be an adequate model* to theorize the ordinary teaching and learning practices. In other words, the "obdurate reality" of classrooms as an empirical field has to be investigated. What kinds of meanings are constructed in students' "ordinary" learning experience? How does the teacher support them? What kind of common ground is being built for the whole class and how does it fit with the cultural definition of knowledge? What do we know about the way teachers select, structure, refine and adjust instructional settings? ...etc. Such questions arose from empirical observations of classrooms at primary school mainly and with an increasing demand for professionalizing teacher education. The institutional location of researches in didactics in teacher training institutes (IUFM in France, since the early 90's) and/or in some department of educational sciences (e.g. Geneva) has broadened the scientific scope of the subject matter didactics toward a comprehensive account of the didactical phenomena as an educational matter. The realm of studies of the didactics of mathematics as a science meet the opportunity to grow from a merely epistemological programme to a quest for an account of human practices that are specified by the conveyance of a socio-historically built culture. In this context, the *paradigmatic research questions of the JATD are new ones compared those featured by the ATD and the TDSM.* The teacher and the students cannot be regarded any longer as epistemic instances merely subjected to the structure of knowledge. The interpretative part of their activity within the educational institutions as a social framework has to be accounted for too. To be clear, we are not arguing that the JATD could replace the fields of investigation that are at the focus of the ATD and/or the TSDM. We would like to point out that it is a complementary framework aiming at giving a status to the subjects' actions and interpretations relatively to the institutional contexts for teaching and learning a given subject matter.

In producing such a framework, we call in some principles that are rooted in both human activity as primarily social and historically built *and* in a pragmatist view of the situations in which the activity develop. Against this background, *the transposition process* sketched by Chevallard and *the didactical contract* theorized by Brousseau, can be viewed as the starting point of *a hybridizing plot*.

First, we postulate that the interpretation of classroom events cannot be performed by focusing solely on either the teacher's actions or the students' ones. We propose to look at *the teacher and students "joint" action* to account for both the historical and the situated interdependence of the classroom actions. Such a joint action may involve separate and distinctive acts that are bound together to make the collective action progressing in some cooperative patterns. The genesis of joint action is based partially on orderly, fixed and repetitious definitions of previous acts through the collective memory that is relayed by the use of signs (graphical, gestual, or vocal). Of course, such joint action is also open to uncertainty and so the transformation of the use of signs to sort new tasks and problems. These statements are general to many actions in human activity (Clark, 1996). A way of specifying them is to consider both the specific purposes of educational institutions and the forms of knowledge to be taught.

i) From TDSM, the didactical contract is probably the most likely principle to address the problem of the individuals' interpretation of contextual practices. We consider that the intention to teach a given topic supported by the teacher generates an expectation to learn "something" from the students. Regularities in the functioning of the classroom as a didactical institution progressively makes the students aware that a teacher usually has "something" in mind beyond the concrete tasks or questions they have to sort. On his/her side, the teacher organises didactical time slots for making the students develop a reflection, an inquiry, the achievement of a task…etc. As soon the student is aware of what is being taught, he/she supposed to know, and the teachers moves on toward another topic. Therefore, teachers and students always remain in an asymmetrical relationship due to the difference in the respective status of their knowledge. We consider *the cultural stance of the didactical contract* as a system of reciprocal expectations merely, according to which the teacher and the students adjust their actions. The asymmetrical status of the teacher and the students relative to their respective relationship to knowledge is consubstantial to the chronogenesis and topogenesis processes that were initially sketched by Chevallard (1985/1991) to describe the structure of recontextualisation of knowledge in the classroom.

ii) However, we do not maintain the constructivist stance of the didactical contract, *i.e.* the contract as regulating an antagonist set of objects that would constrain the students' actions. A converting plot is then required to describe the relationships of the participants to the objects partaking to the situation. Following Mead's definition of the social act (Mead, 1934), we consider that individuals indicate the objects to themselves in line with the function these objects have in collective practices. The meaning-making process is supported by actions –gestures and discourses- in communicative situations. Objects have a meaning for one-self only because they have also have a meaning for otherselves in the situation but also in the culture pre-existing to the situation. Such processes, as indications of objects within the background of language games (Wittgenstein) are actually under investigation for describing the articulation of collective practices and meanings made by individuals. The distinction of "which object counts for which participant", or "from whom this kind of relation comes out" and "who grasps it" is important in determining 1) the set of objects that participants indicate to themselves, 2) the meaning that they may ascribe to their own actions with these objects, 3) the control they gain from it and that may be re-allocated in further experiences. This threefold meaning-making process over time is described as *a mesogenesis*.

iii) Then, it follows that the topogenesis and the chronogenesis are strongly related to the teacher's actions because of his/her leadership in the didactical relation. The teacher is the one supposed to orient the student's actions in order to help him/her learn, but also to notice the student's elaborations in order to designate them a new knowledge. Therefore, some *chronogenetic and topogenetic techniques contribute to the building of a common reference* (objects, relations) in the mesogenetic process. Chronogenetic techniques are anything that the teacher may do in order to *orient the students' actions* toward the piece of knowledge to be learnt. The topogenetic techniques are anything that the teacher does to *regulate his/her involvement* in the joint action and to assign a role to the students all together or as individuals. The devolution and institutionalisation categories for the teacher's action primarily exist in Brousseau's didactical situations, but they may be revised as generic to any teaching process.

iv) The specification of the joint action also operates through the epistemic tasks that are to be achieved. The pre-existing culture necessarily comes in when studying how knowledge to be taught is presented in the teaching materials and curriculum texts. But the purposes of the ordinary practices in classrooms may be rooted in some multi-determination levels other than merely mathematical ones. Thus, acknowledging for the individuals' interpretations of the situations they encounter lead us to reconsider the transposition of knowledge within the didactical institutions from *a bottom-up* point of view that is coupled with the *top-down* analyses typically performed by the ATD framework. We conduct an analysis of the epistemic tasks that are embodied in the teaching materials that the teacher uses (Ligozat & Mercier, 2007). For instance, from the worksheet proposed by the teacher to the students, we may inquire 1) what could be learnt in performing it and then 2) what could be taught according to the curriculum of a given grade. At this step, the fundamental mathematical situations or the mathematical praxeologies provide some useful ways of modelling the epistemic knowledge. The possible gaps and contradictions that are issued by the decontextualisation process may be disclosed against the background of the mathematical practices. Then a bottom up process aims at reconstructing the meanings that objects, situations and practices may have for the participants to the classroom joint action. In this second process, the epistemic model of mathematical knowledge is used as reference to understand 1) what is actually taught and learnt in the joint actions; 2) what the distance left toward the cultural knowledge is and 3) what the epistemic necessities that bend the joint action in some specific ways are. This type of analysis may be carried out at various scales of analyses (a classroom episode, a whole lesson, a teaching unit spread over several lessons…etc.) that can be nested together. The coupling of both the transposition and the social transactions analysis with the classroom supports the investigation method in the JATD framework. A full study of the course of joint actions in the classroom against the transposition of measurement at primary school was achieved in Ligozat (2008).

CONCLUSION

The JATD attempts to encompass a huge programme for didactics as a scientific domain studying the human transactions organised about the transmission of a socio-historically built culture. The need for a theory that aims at theorising teaching and learning practices as they occur in ordinary classroom seems unavoidable. However, in its present state, the JATD has to face different kinds of problems: 1) defining its *identity* as a generic theory for the study of the didactical facts but which develops and produces results by accounting for the specificity of knowledge domains; 2) the further clarification of *its epistemological stances* with respect to the principles and concepts that are borrowed from other theories and 3) the definition of some methodological *units* from its very extended realm of reality, that may be worked out independently without taking the risk of generating some misleading interpretations. The very intention of this paper can be regarded as an attempt to contribute to the first and second points with respect to relationships the JATD has with other theories concerning specific domain didactics. However the clarification of the epistemological stances of the action theories that we invoke still remains a major stake for the works in progress.

References

Arzarello, F, Bosch, M., Gascon, J. & Sabena, C. (2008). The ostensive dimension through the lenses of two didactic approaches. *ZDM Mathematics Education, 40* [special issue], 179-188.

Brousseau, G. (1997). *The theory of didactic situations in mathematics.* Dortrecht : Kluwer.

Chevallard, Y. (1985/1991) *La transposition didactique. Du savoir savant au savoir enseigné*. Grenoble : La Pensée Sauvage éditions.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques, 12*(1), 73-112.

Clark, H. (1996) *Using language*. Cambridge University Press

Douglas, M. (1986). *How do institutions think*. Syracuse University Press.

Forget, A. & Schubauer-Leoni, M.-L (2008). Inventer un code de désignation d'objets au début de la forme scolaire. Des productions personnelles à la convention collective. In L. Fillietaz & M.-L. Schubauer-Leoni, *Processus interactionnels et situations éducatives.* (pp183-204). Bruxelles : De Boeck Université.

Ligozat, F. (2008). *Un point de vue de didactique comparée sur la classe de mathématiques. Etude de l'action conjointe du professeur et des élèves dans le cas de l'enseignement / apprentissage de la mesure à l'école primaire, dans les contextes français et suisses romand.* Thèse de doctorat en cotutelle, Université de Genève et Université de Provence.

Ligozat, F & Mercier, A (2007) The "a priori" inquiry for the study of teacher and student joint didactical action. Oral Communication. Congress of the European Educational Research Association (EERA / ECER), Network 27 : Teaching & Learning / Didactics. 18-20 sept. Ghent, Belgium.

Ligozat, F. & Leutenegger, F. (2008). La construction de la référence dans l'action conjointe du professeur et des élèves. Le cas de milieux différentiels émergeant dans un problème d'agrandissement de distances. *Recherches en Didactique des Mathématiques, 28*(3), pp 319-378.

Mead, G.H. (1934). *Mind, self and society from the standpoint of a social behaviorist.* University of Chicago Press.

Radford, L. (2008) Connecting theories in mathematics education: challenges and possibilities. *ZDM Mathematics Education, 40*, 317-327.

Schubauer-Leoni, M.-L. & Leutenegger, F., (2005). Une relecture des phénomènes transpositifs à la lumière de la didactique comparée. *Revue Suisse des Sciences de l'Education, 27* (3), 407-429.

Sensevy, G. (2009). *Outline of a joint action theory in didactics.* Paper presented to WG9, CERME 6, Jan.28th – Feb.1st 2009.

TEACHER'S DIDACTICAL VARIABILITY AND ITS ROLE IN MATHEMATICS EDUCATION

Jarmila Novotná[1)] and Bernard Sarrazy[2)]

[1)] Charles University in Prague, Faculty of Education, Czech Republic

[2)] LACES-DAESL, Université Victor Segalen Bordeaux 2, France

We are looking for the explanation of the differences in learners' flexibility when using the learned knowledge in new contexts. The main aim of our contribution is to combine various theoretical perspectives of investigating teachers' variability and students' flexibility when applying the learned knowledge. We consider the interpersonal differences as an effect of the teacher's didactical variability. Sarrazy (2002) claims that the question of the use of algorithms and taught theorems by students is more an anthropological than psychological problem. The contribution relates to the question B2: Do different frameworks make us look at different aspects of the learning process, that is, at different research questions and different data, or at different interpretations of the same data about the learning process?

1 INTRODUCTION

Learning mathematics is successful only when the learner is able to identify conditions for the use of knowledge in new situations. These conditions, however, are not present in the algorithms itself and cannot be carried over by teachers to their learners. This is one of the didactical contract paradoxes: "The more the teacher gives in to her demands and reveals whatever the student wants, and the more she tells her precisely what she must do, the more she risks losing her chance of obtaining the learning which she is in fact aiming for." (Brousseau, 1997, p. 41).

In (Novotna, Sarrazy, 2005) we presented two studies originally carried out as independent entities both dealing with the same topic: problem solving. One of them belonged more to the psychological perspective while the second one examined the effects of variability in the formulation of problem assignments on students' flexibility when using taught algorithms in new situations; the research was developed in the framework of the theory of didactical situations. These two studies proved themselves to be perfectly complementary. The first one allowed the detection of a set of phenomena, whereas the second gave them precision through an action model of the problem focusing on the variability in word problems. Connecting these two approaches allowed opening interesting perspectives for a better understanding of the role of problem solving in teaching and learning mathematics by giving precision to certain conditions of their use.

Why is it worth to combine the two approaches? Novotná (2003) showed that the analysis of models created by students enables the teacher to help them in case that their effort to solve the problem correctly is not successful (mainly in determining the type of obstacles the student has faced). The individual differences in the form of graphical models could be explained by the internal student's cognitive processes (Novotná, 1999). However, this approach did not enable us to explain the striking difference "spontaneity versus copying" in the student groups. The psychological perspective did not offer any explanation of the fact observed. It was to be searched for outside the psychological approach. A suitable tool for the explanation was found in the frame of the Theory of didactical situations by Brousseau (1997), namely in the of variability of teachers introduced by Sarrazy (see Part 3).

Sarrazy (2002) presents a model based on the following idea: The more versions of realisations a particular form includes, the more uncertainty is attached to this form. To satisfy the teacher's expectations, the student must 'examine' the domain of validity of his/her knowledge much deeper than a student who is exposed to strongly ritualised (repetitive) teaching and therefore considerably reduced variability.

2 INTERPRETATION OF EFFECTS OF VARIABILITY

We are investigating effects of variability of teachers on learners' flexibility in applying algorithms from three perspectives (for more details see Novotná, Sarrazy, to be published):

a1 – *Psychological interpretation*: Variability gives priority to the change of learners' operational register by diversifying their relationship to the object of teaching or to their action (Richelle, 1986; Drévillon, 1980). In fact, the diversity of modes of relationship to the object of teaching, which is typical for didactical environments with strong variability, brings in an alternation between the phases of knowledge integration and differentiation in their usage. Drévillon (1980, p. 336) states that learners would possess a plurality in their access to objects that would be efficient to help "not only to proceed to the operational formal stage but to construct a repertoire of cognitive registers. This repertoire enables, if asked or needed, to examine a problem and solve it at the functional level, i.e., practical and objective, or to extract the operational quintessence and thus to construct a more general activity model"[1].

According to Piaget (1975, 1981), it is also possible to consider variability as one of the sources of perturbations resulting from variations of didactical environments; this variability enables to provoke cognitive adaptations (accommodations) and thus to increase the student's cognitive register in relation to a conceptual field – e.g., additive and multiplicative structures studied by Vergnaud (1979, 1982, 1994).

This first aspect can be précised didactically by changing the frameworks as proposed by Douady (1986) in the theory of "dialectic 'tool-object' (outil-objet)": "A student possesses mathematics knowledge if he/she is able to provoke its functioning as explicit tools in problems he/she must solve [...] if he/she is able to adapt it when the normal conditions of its use are not exactly satisfying for interpreting problems or for posing questions with regards to it"[2] (Douady, 1986, p. 11).

a2 – *Anthropological interpretation*: When interpreting variability effects in relationship to what could be called "school culture" of the class, variability creates a characteristic of the environment in which learners develop and learn mathematics. In case of weak variability, a repetitive teaching, poorly varying in its forms of organisation and in the content, leads the learners to a hyper-adaptation to proposed situations. In order to adapt themselves to the usual teacher's demands, the learners develop strategies of *coping* (Woods, 1990) with the criteria usually used. They can easily detect indicators allowing them to adapt their decisions and their behaviour to their teacher's didactical requests. In that case, learners can very well apply suitable behaviour without exactly understanding the sense of the lesson or of the problem they were assigned. In case of strong variability, the learners cannot rely solely on the "rituals" because they can neither anticipate nor manage the succession of sequences or behaviours expected by the teacher. The learners' engagement in the situation is much more probable.

It is well known that a particular teacher's attitudes create educational environment, let us call it climate. Flanders (1966) showed the influence of teachers' ways of functioning on the class climate. This climate was defined as "common attitudes that learners have, in spite of their individual differences, with respect to the teacher and the class". In individual cases, this climate can support or block learners' future successful development of their relation towards learning. Certain works in the domain of didactics of mathematics, e.g., Perrin-Glorian (1993) or Noirfalise (1986) support the previous interpretation.

The authors observe that some teachers focus their teaching more on the content to be taught while others on their learners privileging the relationship with the student. The first mainly look for progress in the subject matter and gaining new knowledge, they appreciate all attitudes with which the learners manifest their interest in what they are taught; the latter prefer production of ideas and communication among students. Achievements obtained by students differ significantly according to the considered domains: focus on the content favours success in algebra while focus on the students leads to better results in geometry and to making mathematics more attractive for the student.

a3 – *Didactical interpretation*: As mentioned in a1, Douady's results (1986) allow clarifying the processes enabling to report on the effects of variability. This research is done in two frameworks: Theory of conceptual fields by Vergnaud (1990) and Theory of didactical situations by Brousseau (1997). For Douady, teaching a mathematical concept requires a transformation, a completion to see even the rejection of learners' previous knowledge. The proposed problems must be perceived in such a way that the learners have an opportunity to engage at least one basic solving strategy but this strategy is insufficient: the taught knowledge (*object*) must correspond to the *tool* best adapted to the problem.

Douady distinguishes 6 different phases constituting the process of the “dialectic tool-object”:

Phase a – Mobilisation of “former”: Corresponds to the phase of the problem adaptation by the student.

Phase b – “Research”: Corresponds to the phase of action of the Theory of didactical situations (Brousseau, 1997). During this phase, students encounter difficulties caused by the insufficiency of their previous knowledge and consequently look for new, better adapted instruments.

Phase c – “Local explication and institutionalisation”: The teacher points out the elements that played an important role in the initial phase and formulates them in terms of the object with the condition of their use at the given moment.

Phase d – “Institutionalisation” (in the sense of the Theory of didactical situation by Brousseau, 1997): The teacher gives a cultural (mathematical) status to the new knowledge and he/she requests memorization of current conventions. He/she structures the definitions, theorems, proofs, pointing out what is fundamental and what is secondary.

Phase e – “Familiarisation - reinvestment”: It concerns the maintenance of what was learned and institutionalised in the various exercises.

Phase f – “Complexification of the task or a new problem”: The aim of this last phase is to allow the students to make use of the new knowledge in order to allow new objects to occupy their position in the students’ previous knowledge repertoire.

According to Douady, the aim is to exploit the fact that most mathematical concepts operate in several frameworks – in fact in diverse types of problems. For example, a numerical function can be presented at least in three frameworks: numerical, algebraic, and geometrical. These changes of frameworks (“game of frameworks”) allow varying the significances (supports of significations) for the same concept and allow avoiding that the learners make them function in a partial or in over-contextualised ways. The interactions among diverse frameworks allow, according to Douady, to make the knowledge progress and to keep all the conceptual potential of the taught object.

3 EXAMPLE: SARRAZY’S MODEL OF TEACHERS’ VARIABILITY

For the characterisation of teachers’ modes of didactical activity, typology of modes and examination whether these modes enabled awareness of the differences in the sensitivity to didactical contract in groups of students, Sarrazy (1996) introduced a model that allows describing the modes of teachers’ actions. This model is sensitive in learners’ treating of problem types. It uses the following three dimensions, the six variables being defined in order to measure variability in organisation and management of the teacher’s work during and between lessons:

i) *Didactical structure of the lesson* (what the teacher really does from the perspective of the knowledge to be taught);

v_1. What is the type of didactical dependence? Does the teacher proceed from simple to more complex tasks or the other way round?

v_2. Place of institutionalisation: At which moment does the teacher present a solving model? Closer to the beginning or to the end of the lesson? Or only at the beginning or at the end?

v_3. Types of validation: How are the students informed about validity of their answers? Does the teacher always use the same type of evaluation and assessment (by the milieu, by direct evaluation, by the Topaze effect[3], by peers …)?

ii) *Forms of social organisation* (this domain corresponds to the teacher's activities regarding class management)

v_4. Interaction modes: teacher-student(s), student(s)-student(s) … .

v_5. Management with regard to the students' groupings: the whole class, small groups, individual work … .

iii) *Variability of arithmetical problem assignment*

v_6. The variable is related to editing the problem assignment. It is given by an indicator which measures the teacher´s "capacity" to consider diverse modalities of the same didactical variable in the assignment.

This model makes it possible to describe the teacher's teaching practices from a triple perspective: presentation of the content (i), desired forms of teaching (ii) and variety of the proposed situations (iii). It is not an isolated variable that affects the students' learning (mainly defined by the notion of sensibility – i.e. their ability to use the taught algorithms in various contexts). On the contrary, it is an effect linked to a set of variables (that may be called a profile of the didactical action); this profile enables a characterization of one way of letting the students do mathematics. This is why we proceeded to a hierarchical classification in order to show similarities by clustering of variables.

Using the above variables, teachers' different profiles were hierarchically classified (Sarrazy, Novotná, 2005, where the experimental disposition, that allowed characterising teacher's variability and thence to show the influence on the way how the students do mathematics, is presented; the crucial role of didactical contract and the sensitivity to it is documented).

Let us recall here the general idea: Submission of students to a teaching style poorly varied (and strongly repetitive in the forms of organisation in the presentation of the content) will decrease the possibilities of opening the didactical contract; vice versa, more variable the teaching is, the more the students will be confronted with new situations and the more flexible their use of the taught algorithms will be. Let us consider a simple (and therefore caricaturing) example which serves as an illustration of the theoretical position:

> The mother spent 13 EUR at the market. Now, she has 19 EUR. How much had she had when she went to the market?

This problem, although simple, presents several difficulties to the students. These difficulties are based on the fact that the problem evokes the framework of subtraction but the numerical operation to be executed is addition. Here is an example of the variety: the more the student will be confronted with the situations that involve divers contexts of the use of additive structures, the higher the probability that his/her answer will be guided by conceptualising the relations in play; vice versa, the less diverse the situations are, the more the students will be lead to rely on the apparent characteristics of the tasks when producing their answer (e.g.: every time seeing the verb "spent" they will subtract, "anybody" divide etc.).

Using the above variables we defined three teaching styles of the school culture that are in strong contrast:

"*Devolving*": This style corresponds to what, in the first approximation, could be called "active pedagogy" in which the students need to be "active". This style is characterised by strong variability in the organisation and management of situations: the teachers regularly use group work although they by no means restrict only to this form of student work; generally speaking, the problems are complex; classroom work is very interactive (students interact spontaneously, "choral" answers are not rare, …); in the lesson, institutionalisation is diverse. These are the main features of the first style.

The other extreme is the "*institutionalising*" style. This climate is characterised by a weak introduction and a weak variety of situations presented to students; we could call it 'classic teaching' in which the scheme "show–remember–apply" seems to be the rule. These teachers institutionalise one solving model very quickly and then present students with exercises of growing complexity. First, the exercises are corrected locally – the teacher passes through the rows and corrects them individually. Then the teacher gives the complete correction on the blackboard; here he/she gives details of the solution and, depending on the time he/she has, occasionally invites some students to the board either to make sure that they are paying attention, or to recall certain knowledge. Now, the interactive climate is quantitatively as well as qualitatively very different from the interactive climate of the preceding style: Students' spontaneous interactions or "choral" answers hardly ever occur.

The third style is the "*intermediary*" style. As its name indicates, this style is closer to the institutionalising style, even if the teachers 'open' the situations more and more frequently. In any case, here the students have more chances than students of "institutionalising" teachers to encounter research situations, and debate, but markedly less than those exposed to the "devolving" style.

As we expected, we observed strong internal coherence of each of the styles (climates) confirmed by the stability of the results acquired using various methods of data analysis (implicative analysis, dynamic clusters, hierarchical classification, and so on). It seems to provide evidence in favour of the existence of an organising principle for the practices. This organising principle could at the same time be linked with didactical conditions (meant in relation to the knowledge dealt with) and with anthropological conditions (independent of knowledge but linked with teachers' pedagogical or political convictions, with influences of fashionable constructivist, cognitive, and other psychological models).

4 CONCLUDING REMARKS

There are two concluding topics to be discussed: the consequences of the presented results for teacher training and the theoretical positions of the studies about variability.

The presented results are of great interest for improving the teaching of mathematics by focusing on the flexibility in the use of the taught algorithms. But is it possible to foster an increase in the variability of the teachers? It seems to be difficult to directly influence the conditions allowing increasing the variability of teachers. Even if we find it important to present teachers with models of the analysis of problem assignments (e.g. those of Vergnaud concerning additive and multiplicative structures), there are good reasons to believe that mere presentation is not sufficient. In fact, on the one hand these models when only presented to teacher trainees to have a look at them do not affect their variability directly (Sarrazy, 2002); on the other hand, we could observe that variability is the dimension of the teacher's activities that is statistically linked with other dimensions of his/her didactical activities (e.g. the use of group work, the volume of didactical interactions, his/her pedagogical philosophy). Variability should be understood as one of the elements of the teacher's system of didactical activities that interacts with other components.

This last aspect bids for discussion of its theoretical status. We do not pretend to submit here a new theoretical concept of a teacher's didactical activity but more modestly, we situate this approach as an extension of the Theory of Didactical Situations by Brousseau (1997). During the "ordinary" teaching situations that we observed, we found rarely those where the "milieu" contained an a-didactical component, i.e. those where the situation allowed to delegate to students the retroaction to their actions. We believe that a developed variability when the a-didactical "inside" of the situation is absent, would allow the students to establish a quasi a-didactical relation only. As we indicated, it is the consequence of the fact that they cannot go upon the formal aspects of the proposed assignments.

An important question arising from our research is: What kind of training is likely to increase the variability of teachers? Although it is certainly an important question, we find solving it premature as long as the problem of conditions favouring the variability has not been clarified. This problem, first opened in anthropo-didactical approach in DAESL about fifteen years ago, needs to be explored in further research in the area where didactics and pedagogy meet.

REFERENCES

Brousseau, G. (1997). *Theory of Didactical situations in mathematics 1970-1990*, [Edited and translated M. Cooper, N. Balacheff, R. Sutherland and V. Warfield.] Dordrecht: Kluwer Academic Publishers. (French version 1998. *Théorie des situations didactiques*. [Textes rassemblés et préparés par N. Balacheff, M. Cooper, R. Sutherland, V. Warfield]. Grenoble: La pensée sauvage.)

Douady, R. (1986). Jeux de cadres et dialectique outil-objet. *Recherche en Didactique des Mathématiques*. La Pensée Sauvage, Volume 7-2.

Drévillon, J. (1980). *Pratiques éducatives et développement de la pensée opératoire*. Paris: PUF.

Flanders, N.A. (1966). *Interaction analysis in the classroom: A manual for observers*. Michigan: Michigan School of Education.

Noirfalise, R. (1986). Attitudes du maître et résultats scolaires en mathématiques. *Recherches en Didactique des Mathématiques, 7(3)*, 75-112.

Novotná, J. (1999). Pictorial representation as a means of grasping word problem structures. *Psychology of Mathematical Education*. Retrieved December 7, 2008, from http://people.exeter.ac.uk/PErnest/pome12/default.htm.

Novotná, J. (2003). *Etude de la résolution des « problèmes verbaux » dans l'enseignement des mathématiques. De l'analyse atomique à l'analyse des situations*. Bordeaux: Université Victor Segalen Bordeaux 2.

Novotná, J. & Sarrazy, B. (2005). Model of a professor's didactical action in mathematics education. Professor's variability and students' algorithmic flexibility in solving arithmetical problem. In J.-P. Drouhard (Ed.), *CERME 4*, WG 6. Sant Feliu de Guíxols, Španělsko, 17.-21.2.2005. Retrieved December 7, 2008, from http://cerme4.crm.es/Papers%20definitius/6/NovotnaSarrazy.pdf

Novotná, J. & Sarrazy, B. (in preparation for print). Didactical variability in teacher education. In O. Zaslavsky & P. Sullivan (Eds.), *Constructing knowledge for teaching secondary mathematics: Tasks to enhance prospective and practicing teacher learning*. Springer.

Perrin-Glorian, M.-J. (1993). Questions didactiques soulevées à partir de l'enseignement des mathématiques dans des classes 'faibles'. *Recherches en Didactique des Mathématiques*, 13(12), 5-118.

Piaget, J. (1975). *L'équilibration des structures cognitives*: *Problème central du développement*. Paris: PUF.

Piaget, J. (1981). Creativity. [Transl. by E. Duckworth.] In J. M. Gallagher & D. K. Reid (Eds.), *The learning theory of Piaget and Inhelder* (pp. 221-229). Monterey CA: Brooks Cole.

Richelle, M. (1986). Apprentissage et enseignement: réflexion sur une complémentarité. In M. Crahay & D. Lafontaine (Eds.), *L'art et la science de l'enseignement* (pp. 233-249). Bruxelles: Labor.

Sarrazy, B. (1996). *La sensibilité au contrat didactique*: *Rôle des Arrière-plans dans la résolution de problèmes d'arithmétique au cycle trois*. [Thèse pour le doctorat de l'Université de Bordeaux 2 – Mention Sciences de l'Education.] Bordeaux: l'Université de Bordeaux 2.

Sarrazy, B. (2002). Effects of variability on responsiveness to the didactic contract in problem-solving among pupils of 9-10 years. *European Journal of Psychology of Education*. XVII(4), 321-341.

Sarrazy, B. & Novotná, J. (2005). Didactical contract: Theoretical frame for the analysis of phenomena of teaching mathematics. In J. Novotná (Ed.), *SEMT 05* (pp. 33-45). Praha: Univerzita Karlova v Praze, Pedagogická fakulta.

Vergnaud, G. (1979). The acquisition of arithmetical concepts. *Educational studies in Mathematics, 10*, 263-274.

Vergnaud, G. (1982). Cognitive and Developmental Psychology and Research in Mathematics Education: some theoretical and methodological issues. *For the learning of Mathematics*, *3(2)*, 31-41.

Vergnaud, G., (1990). La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques*, *10(2-3)*, 133-170.

Vergnaud, G. (1994). Multiplicative Conceptual Field. What and Why. In G. Harel & J. Confrey (Eds). *The Development of Multiplicative Reasoning in the learning of Mathematics*. Albany State: University of New York Press.

Woods, P. (1990). *The Happiest Days?: How Pupils Cope With Schools*. Taylor & Francis.

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[1] Translation from French by J. Novotná. Original text: « non pas seulement à passer au stade opératoire formel mais à construire un clavier de registres cognitifs. Ce clavier permet à la demande, et en cas de besoin, d'examiner un problème et de le résoudre au niveau fonctionnel, c'est-à-dire pratique et objectif, ou d'en extraire la quintessence opératoire et de construire ainsi un modèle plus général de l'activité. »

[2] Original text: « Un élève a des connaissances en mathématiques s'il est capable d'en provoquer le fonctionnement comme outils explicites dans des problèmes qu'il doit résoudre […] s'il est capable de les adapter lorsque les conditions habituelles d'emploi ne sont pas exactement satisfaites pour interpréter des problèmes ou poser des questions à leurs propos ».

[3] ***Topaze effect.*** When the teacher wants the pupils to be active (find themselves an answer) and when they can’t, then the teacher suggests disguises the expected answer or performance by different behaviours or attitudes without providing it directly. Example: Teacher: 6 x 7? Pupils: 56. Teacher: Are you sure?

THE POTENTIAL TO ACT FOR LOW ACHIEVING STUDENTS AS AN EXAMPLE OF COMBINING USE OF DIFFERENT THEORIES

Ingolf Schäfer

University of Bremen

In dealing with low achieving students one needs a fine grained measure for their gain in knowledge. I will show that the concept "potential to act" helps to understand the students' difficulties and to support their construction of knowledge. The concept connects parts of theories of different scope: a model for abstraction in context, self-determination theory and a psychological theory of action. The relevant parts of the theories will be discussed, and, more specifically, to which extend they are compatible. I shall utilize an example to illustrate the concept of the "potential to act" and to show the interplay of the different theories at work. Further, I will explain how their combining use gives rise to additional insight about the construction of knowledge.

INTRODUCTION

As part of an on-going project at the mathematics education department of the University of Bremen, I am working on a theory of support for low achieving students in Hauptschule[1], aged between 13 and 18. In the project, we want to identify what kind of potential to act in certain situations these students have in order to be able to adapt the supporting lessons better to them, and to understand how they construct and reconstruct mathematical knowledge. For this it is necessary to get finer information about the students' gain of knowledge than is possible by error analysis of direct tasks or questionnaires.

We are not discussing the phenomenon of low-achieving students in terms of "dyscalculia" or similar notions (cf. (Moser Opitz, 2007) for a recent review). Those studies concentrate mainly on primary school students and on typical problems with arithmetic and numeracy tasks. In contrast, I am interested in the problems of motivation for low-achieving students, which seem to have gained little interest so far. A notable exception is the article of Pendlington (2006), where the author describes the effect of supporting lessons on self-esteem.

I will not use the concept of self-esteem in this paper, but I will make use of self-determination theory for the motivational aspect. Furthermore, I complement this approach with the theory of abstraction in context and a theory of action. By applying these parts of different theories we can accomplish a more complete understanding of the learning process for low-achieving students.

In this paper I present a case of combining three different theories that in their cores may not be fully compatible and this case raises the question what compatibility means in this context.

THEORETICAL BACKGROUND

I will restrict the description of the three theories to their main parts.

Abstraction in context – the RBC model

Hershkowitz, Schwarz & Dreyfus (2001, p. 202) regard abstraction as “an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure”. This means that abstraction is an activity in the sense of Leont'ew's activity theory that comprises actions. Hershkowitz et al. identify three characteristic epistemic actions, namely recognising (R), building-with (B), and constructing (C). Recognising is described as an action in which a student becomes aware of a familiar mathematical structure in the situation, and building-with as “combining structural elements to achieve a given goal” without gaining new complex knowledge about the situation. When this happens constructing takes place.

These epistemic actions are observable in social interaction and provide evidence that a process of abstraction is taking place. The actions are nested, e.g. constructing requires that the subject has already recognised and built with existing structures to construct a new mental structure.

Self-determination theory (SDT)

The self-determination theory (Deci & Ryan, 2000; Ryan & Deci, 2000b) explains how different kinds of motivation emerge. For this the existence of three innate psychological needs is postulated: the need for autonomy, the need for competence and the need for social relatedness. These needs “specify the necessary conditions for psychological health or well-being” (Deci & Ryan, 2000, p. 229) and are indispensable for intrinsic motivation or integration of extrinsic motivation. Following Bikner-Ahsbahs (2005) I specify the innate needs for students in mathematics as follows: *autonomy* as the experience of being able to initiate learning processes and decide about them, *relatedness* as the experience of integration in the social environment and of social support. Bikner-Ahsbahs’s definition of *competence* as experience of broadening or deepening one’s mathematical abilities seems to be too narrow for our purpose, because low achieving students might get a feeling of competence simply by successful application or reproduction of their mathematical knowledge.

Theory of action

Oerter (1982) discusses the notion of action and the relation of objects and action. He follows the tradition of Leont'ew's activity theory and considers action to be of “primary reality” for each subject, i.e. action is the sole link between an individual and its environment.

> “There is no remembering, imagining or thinking as such, other than with respect to the objects of the environment.” (Oerter, 1982, p. 103, transl. by the author)

This implies that any kind of relation to objects or between different objects can only be accomplished by action. There are three layers of object relations[2].

1. *no separate object*, i.e. the object is a fixed part of the situation and cannot be thought of after the current action. It will not even be recognised as an object.
2. *object separated from subject*, i.e. a relation beyond the current action. A subject can recognise the object and name it after the current action but it may still be dependent on the given situational context.
3. *abstract, formal object*, i.e. the common structure of the contextualized objects.

Our experiences with low-achieving students lead to the hypothesis that these students often fail at the transitions from one level to the other. For example, let us take a quarter of a certain cake. At the first level, the student does not realize a separate object at all, i.e. this quarter has no meaning by itself and after it has been eaten there is nothing left to think about. At the next level, the meaning of a quarter of this cake can be transferred to similar situations. So, we might think of a quarter of a piece of chocolate, but all of those quarters are still tied to their context. Finally, at level three a student might have a concept of a quarter of something, meaning one of four equal parts of an entity. Thus, this concept has become abstract and does not depend on the concrete action.

THE POTENTIAL TO ACT

We start with the definition: *The potential to act* consists of all possibilities a subject has to act in a given situation with respect to given objects. This rather abstract definition requires some explanation and we shall discuss it in a more concrete setting:

Imagine that you are working with a student on some mathematical concept, e.g. division of natural numbers. Using a traditional test you have already found out that he fails to solve most division tasks. Furthermore, you have experienced that he cannot make use of most basic ideas associated with division of natural numbers. However, if you ask him to explain how something might be divided in a certain family situation, he can explain some of these basic ideas. In this case his potential to act includes these concepts in the family situation, but not in the written test. So, using the family situation, you might be able to help him enlarge the potentials to act for division tasks.

It is obvious that it is impossible to describe the potential to act of a given student completely. Nevertheless, by looking at the real actions (in contrast to the potential ones) a researcher is able to identify indicators for them and can develop hypotheses about how the student’s potential to act might look like in this specific situation and similar ones.

A potential to act can be described by two dimensions: the cognitive dimension and the motivational dimension. The RBC-model and the SDT provide tools to gain indicators in these dimensions. Let us briefly describe what these dimensions mean and how to get indicators for their description.

The motivational dimension is thought of as the degree of intrinsic motivation. Whenever an innate psychological need is satisfied, we interpret this according to SDT as an indicator for an increase in the motivational dimension. If the needs for competence, relatedness or autonomy are not satisfied, we infer that intrinsic motivation will decrease. At this stage of research we use the words *increase* and *decrease* in a qualitative sense without any quantification.

The epistemic actions of the RBC-model may serve as indicators for the cognitive dimension of the potential to act. This dimension inherits the hierarchy of the nested epistemic actions.

Besides the cognitive and motivational dimension, one has to cope with situational aspects of the potential to act including the objects involved. The layers of object relation are used as a tool to structure and categorize the objects in different situations.

Let me briefly comment why those three theories were chosen for the aspects of the potential to act. In order to have a framework for the notion "potential to act", I chose the theory of action according to Oerter, which has the advantage to offer a description of relations to the objects. The theory of abstraction in context is used, because it allows gaining information about the process of construction of knowledge and fits well with Oerter's framework of action. Self-determination theory was chosen, because it captures the motivational aspects of the potential and has already been successfully used in describing the motivational problems of low-achieving students in general (Skinner & Wellborn, 1997).

SOME DATA

The data shown below stems from an explorative study conducted at the University of Bremen to explore the potential to act for a group of low achieving students. The students where of age 14 to 18 and took part in weekly supporting lessons, which were done either for groups of three students or individually. The lessons were videotaped and the video was analyzed afterwards to reconstruct the potential to act and to set up the tasks for the next lesson based on this analysis.

The following transcript shows part of supporting lessons that were intended to help the student (S) to understand the concept of equivalence of fractions. This specific task was chosen to help S to develop connections between different representations of extending fractions. S is 14 years old and has been taught by a special school teacher in mathematics for over a year before she came into our project. In her math class fractions had already been introduced the year before and were again the topic of various lessons in class during the weeks before this episode was conducted. After S has been given a worksheet showing figure 1 the teacher (T) asks her to explain the diagram.

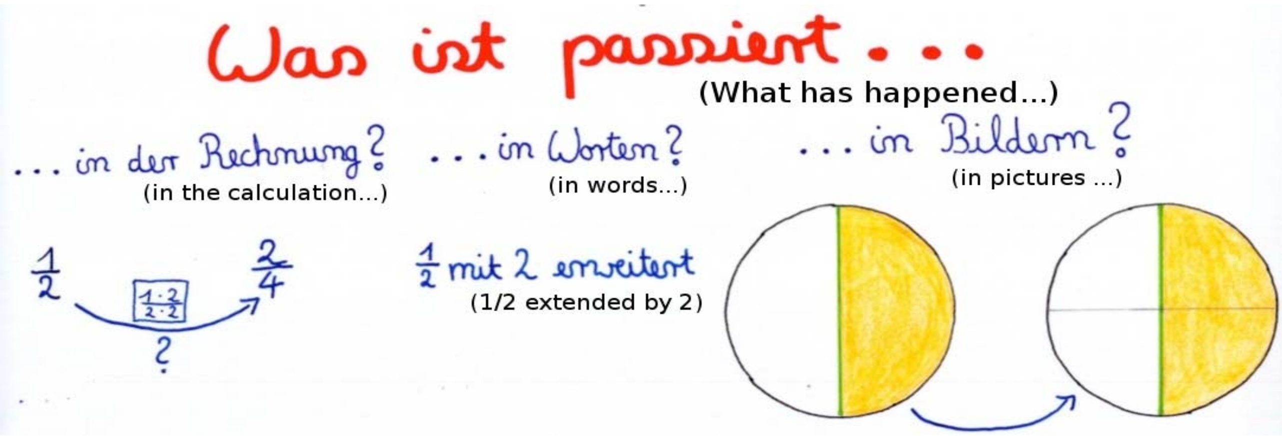

Figure 1: "What has happened …?" (translation by the author)

#	Speaker	
2	S	Well – erm – they have one half – times – they have calculate one times two – up here, haven't they? (S points at calculation in the denominator)
3	T	Hmm.
4	S	And – erm – what then four – erm – to get four as a result, they have calculated two times two.
5	T	Hmm, exactly.
6	S	Well, they have extended by two.
7	T	- And what is this picture?
8	S	Erm, that is one half and … quarter. Two quarters.
9	T	Hmm. And what exactly has this picture to do with – erm – the calculation?
10	S	This is one half and this – and these are two halves. (S points at ½ in calculation and left circle, 2/4 in calculation and right circle in fig. 1)
11	T	Hmm – exactly, fine, and – er – now in here there is this, this calculation described, isn't it? You have said this correctly already. Erm, can you find this, what has happened here, this calculation. Can you find it in here again?
12	S	(S pauses for 17 seconds) one times two is this (S points at left circle in fig. 1) and two times two this (S points at right circle in fig. 1) two and two (S smiles) –
13	T	(T shrugs, then smiles) Erm, two times two – where does it say that?
14	S	Down there.
15	T	Erm. And do you know, what it means, if it is written down there?
16	S	(S pauses for five seconds) If it says 2 times 2 below, then we must do "times two" above.

Transcript 1: "What has happened…?" (translation by the author)

Before we analyze the potential to act, let us first summarize the situation. The lines 11 – 13 point at the crucial situation. The student is asked to explain how the process of extension by two is visualized in the picture. She is expected to say that this is done by refining the given fraction. While the teacher explains to S after line 16, what the answer should have been, S is looking out of the window and seems frustrated. S does not engage herself anymore in the rest of the supporting lesson and is very serious.

We reconstruct S's potential to act in three steps. First, let us consider the epistemic actions. There are a number of recognising actions in lines 2, 4 and 6. S recognises the calculation in the numerator and the denominator of the fraction in the blue box in fig. 1. She also recognises the left circle as a half and the right circle as two quarters (8) and is able to relate them to the corresponding fractions in the calculation. In line 11 she is asked where to find the calculation inside the blue box in the picture. After a short pause, she identifies "one times two" as the left circle, and "two times two" as the right circle. This should be considered as a building-with action, because she puts together the things she has already recognised and she has to think about this question. In line 16 she also builds-with, because she states a general rule for the objects. Unfortunately, we do not know why she thinks this rule is valid.

What about the motivational component in this situation? There is no experience of autonomy in this transcript, because the task is very explicit and she has not been given much choice how to deal with it on her own. But we can see some experiences of competence here. She is able to identify the fractions in lines 8 and 10, and the teacher supports her by saying "exactly" and "fine". This experience of competence is deepened by S's answer to the question in line 11. S thinks for 17 seconds and manages to give an answer that makes her smile; she seems content with her own abilities. But the reaction of the teacher (shrug) and the teacher's later explanations reverse this experience of competence into the opposite. S realizes that her answer was wrong and may feel even more incompetent because she did not manage to understand that this answer was wrong. Likewise the need for relatedness might be fulfilled by the support S gets from the teacher and the smiling, a bit later this support might seem hollow and misleading. In summary, none of the three innate needs is satisfied here.

Using Oerter's layers of object relation we may interpret this episode further. For S the calculation is not one object, but likely she thinks of a pair of objects, i.e. two separate multiplications. Therefore she looks for a corresponding pair of objects that are given by the two circles in fig. 1. She uses the name "extend by 2" only once in line 6 and it may just be, because it is written on the sheet. Given she names the process of extension on her own, then her relation to this process as an object is in the second layer. But she does not even seem to be able to identify this process as an object of its own right (Oerter's first layer). Thus, her relation to the object "extension by 2" is somewhere between the first and second layer. Line 16 indicates that she might actually be closer to the second layer, but we do not know, why S thinks one "must do 'times two' above". We do not know whether she is really able to understand this extension as an object of its own right, i.e. as a process that transforms one fraction into an equivalent one.

In summary, S is involved in the situation up to line 16, recognises and builds-with the corresponding mathematical objects. Her innate psychological needs are satisfied up to here. Since S is not able to identify the calculation in the picture correctly, T starts explaining how to understand the picture after this episode, which leads to the experience of incompetence for S. Using the layers of object relation we argue that S cannot correctly identify the extension process for the circles because she is only partially able to think of the extension by two as an object. Thus, she cannot recognise it or build-with. Moreover, this information in mind future supporting lessons can be planned to foster S in the transferring to the next layer of object relation.

The analysis above demonstrates that the use of only one theoretical perspective is not enough to understand the data in sufficient generality for the given purpose. Using the RBC-model we saw that S built-with the structures she recognised, i.e. she was engaged in the process so far. SDT can explain why her engagement stops and in terms of the layers of object relation we can understand her epistemic problem and why she could not construct or reconstruct the concept of "extension by 2" in the given situation. Leaving out one perspective results in serious loss of information, e.g., if the SDT was left out, we would know the epistemic problem but could not explain the sudden change in S behaviour.

SOME PRELIMINARY FINDINGS

It should be kept in mind that the following results are only some preliminary findings from the explorative study. They should be thought of as hypotheses for a larger study to be tested.

Low achieving students seem to make use of a large repertoire of avoidance strategies in order to cope with given tasks. Especially, if their basic psychological needs were not satisfied the students responded by withdrawal, denial or similar actions, as seen above.

Furthermore, the students' potential to act seems to be very dependent on the situational context. Frequently, their relations to the objects were found to be at the first or second layer, hence, the students had no abstract understanding of the objects. If the object relation was at the first layer, the students were not able to recognise the objects and thus could not do building-with actions. At the second layer students frequently developed different versions of an object depending on the context, e.g., a student had developed two different and unrelated object relations of a hexahedron having only the name in common.

TOWARDS THE USE OF THE DIFFERENT THEORIES

Prediger, Bikner-Ahsbahs & Arzarello (2008) suggest a landscape of strategies for connecting theories, which can be ordered by the degree of integration of theories. I shall now explain where the position of my approach in this landscape is.

I use the three theories as a way to understand the different dimensions and aspects of one concept. In terms of Prediger et al. I combined the different parts here "in order to get a multi-faceted insight into the empirical phenomenon in view" (Prediger et al., 2008, p. 173). It may even be that I coordinated, i.e. developed "a conceptual framework built by well-fitting elements from different theories" (ibid., p. 172). For this "a careful analysis of the mutual relationship between the different elements" is necessary and it "can only be done by theories with compatible cores" (ibid., p. 172). To decide the question whether I combined or coordinated let us consider the relationship of the theories:

From the broadest perspective, we have two psychological theories (SDT and the theory of action) and a theory originated in mathematics education research (RBC). SDT and RBC focus on the individual, Oerter's theory on social interaction, but there is no obvious contradiction at this level between these approaches.

The epistemic actions of the theory of abstraction in context have their roots in activity theory (Pontecorvo & Girardet, 1993). Oerter's concept of action is also motivated by activity theory and as far as foundations and basic assumptions are concerned, both theories are compatible.

How do these theories relate to SDT? SDT is a theory in cognitive psychology and at its core are the three innate psychological needs, which act as inner regulation processes that regulate and determine behaviour:

> "SDT describes and predicts the occurrence of distinct processes by which behavior is determined or regulated, some of which are characterized as autonomous and some as controlled or amotivational. We assume not only that these forms of regulation differ experientially, but they also differ in their antecedents, their consequences, and their neuropsychological underpinnings." (Ryan & Deci, 2000a, p.330)

It seems impossible to express the above quotation from Oerter's point of view. His fundamental critique is that action should not be thought of as an intentional but as the primary concept in psychology (Oerter, 1982, p.102). Every other concept has to be developed based on and connected to action. It is not clear to me, whether this implies contradicting basic assumptions, since the notion of "behaviour" by Deci and Ryan is not compatible with Oerter's actions.

What are the relations between different terms in the theories? The potential to act is a concept defined in the notions of Oerter's framework. The epistemic actions are expressed in terms of activity theory and can be understood in Oerter's framework without any change. The three innate psychological needs are defined through experiences of the subject that are the results of certain actions. Autonomy, for example, was defined as the experience to be able to initiate learning processes and decide about them. This experience is the result of a successful initiation or decision action by the individual itself or by the social group, e.g. the class. In this way the potential to act and all terms used to investigate it can be coherently expressed in terms of the theory of action.

Since the main difference between coordination and combination of theoretical frameworks is whether the theories are compatible, which includes non-contradicting assumptions, I cannot say which one I did, although I have built up a coherent philosophical base above.

SUMMARY AND OUTLOOK

In this paper I presented the definition of the potential to act and applied it to an example using empirical data. It was utilized and helped to gain insight in the process of the construction of knowledge and the motivational aspects of it.

The interplay of the three theoretical parts in the potential to act was described and I tried to position myself into the landscape of connecting theories following Prediger et al. (Prediger et al., 2008).

Bearing in mind the difficulties I had to find the position of my approach, I ask what the meaning of the notions “compatibility of theories” and “non-contradicting cores of theories” is. Does it mean a theory is compatible with another one just because their terms are incommensurable? When do basic assumptions contradict? Cobb (Cobb 2007) remarks that there is no algorithm how to deal with different theoretical perspectives. I suppose that there is also no algorithm to guarantee enough compatibility such that one has not build up “inconsistent theoretical parts without a coherent philosophical base” (Prediger et al., 2008, p. 173), but there might be general strategies which can serve as guide lines for he process of analyzing compatibility.

The “potential to act” is part of my research on low achieving students. The long-term goal is to have a theory of support for low achievers which builds upon the enlargement of the potential to act. A first explorative study has been done on this and my next step is to use the experience gained there in a larger study on support for low achieving students.

NOTES

1. Hauptschule is a secondary school for children, which are supposed to be in the lowest achievement category

2. It should be noted, that these layers are simplified versions of Oerter’s layers adapted for the purpose at hand.

REFERENCES

Bikner-Ahsbahs, A. (2005). *Mathematikinteresse zwischen Subjekt und Situation : Theorie interessendichter Situationen - Baustein für eine mathematikdidaktische Interessentheorie*. Hildesheim: Franzbecker.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3-38). Reston: NCTM.

Deci, E. L., & Ryan, R. M. (2000). The "what" and "why" of goal pursuits: Human needs and the self-determination of behavior. *Psychological Inquiry*, 11(4), 227-268.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, 32(2), 195-222.

Moser Opitz, E. (2007). *Rechenschwäche-Dyskalkulie : Theoretische Klärungen und empirische Studien an betroffenen Schülerinnen und Schülern*. Bern.: Haupt-Verl.

Oerter, R. (1982). Interaktion als Individuum-Umwelt-Bezug. In E. D. Lantermann (Ed.), *Wechselwirkungen* (pp. 101-127). Göttingen: Verlag für Psychologie.

Pendlington, S. (2006). Mathematics isn't easy. *Mathematics Teaching Incorporating Micromath*, 197, 3-8.

Pontecorvo, C., & Girardet, H. (1993). Arguing and reasoning in understanding historical topics. *Cognition and Instruction*, 11(3/4), 365-395.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM*, 40(2), 165-178.

Ryan, R. M., & Deci, E. L. (2000a). The darker and brighter sides of human existence: Basic psychological needs as a unifying concept. *Psychological Inquiry*, 11(4), 319-338.

Ryan, R. M., & Deci, E. L. (2000b). Self-determination theory and the facilitation of intrinsic motivation, social development, and well-being. *American Psychologist*, 55(1), 68-78.

Skinner, E. A., & Wellborn, J. G. (1997). Children’s coping in the Academic Domain. In S. A. Wolchik & I. N. Sandler (Eds.), *Handbook of Children’s Coping. Linking Theory and Intervention* (pp. 387-422). Plenum Press: New York

OUTLINE OF A JOINT ACTION THEORY IN DIDACTICS

Gérard Sensevy

CREAD Université Rennes 2-IUFM de Bretagne/Université de Bretagne Occidentale

ABSTRACT

In this paper my goal consists of presenting aspects of the Joint Action Theory in Didactics on the principle of a twofold specification (Didactic Game and Learning Game), after integrating it in a more general picture. I first make a general presentation of the epistemological background against which the Joint Action Theory in Didactics could be seen. Then the second part of the paper is devoted to the description of a system of tools which constitutes the core of the JATD. In the third part, I give an example of empirical analysis in order to illustrate the categories presented previously. In the last part of the paper, I make some conclusive remarks in order to contribute to the networking process that this group is elaborating.

INTRODUCTION

In this paper I present some aspects of a collective work (Sensevy & Mercier, 2007; Schubauer-Leoni, Leutenegger, & Forget, 2007; Ligozat 2008), which functions as a *collective thought* from which I take most of the ideas I express in this contribution.

1. THE JOINT ACTION THEORY IN DIDACTICS: AN EPISTEMOLOGICAL BACKGROUND

1.1 The logic of practice, language-game and semiosis

In Social Sciences, the main challenge is probably to understand the meaning-making process in practices, thus understand the logic of practice on which people base their behaviors. In our conception, acting according to the logic of a practice is to be able to master a specific language-game in a particular life-form (Wittgenstein, 1953/1997). In order to master this language game, one has to be able to decipher signs of various kinds in an appropriate way. Acting according to the logic of the practice is therefore to be able to participate in a specific *semiosis* process (see Lorenz, 1994). To do that, people have to draw the same conclusions from a given environment, to give the same meaning to the prominent features of this environment. Inside this frame, I argue that the fundamental meaning-making process is an inference process, by which one can grasp and express the logic of the practice, and, doing that, can demonstrate understanding and agency.

1.2 The inference-reference process: institution and thought style

I assume that meaning is mainly processed in analogical inferences. In order to understand how these analogical inferences are made, one must consider that they are processed in context, the analogies being produced from a context to another. A theoretical point is thus to characterize what is a context, that I consider as an *institutional milieu*. Such an institutional milieu can be viewed as a specific reference, a background against which the agreement on inferences ("joint inferences") is made. Language-game mastering, semiosis process, and inference-reference strategies in an institutional milieu gather in the process of *recognition of forms*, which is the central feature of our conception of cognition and language. A way of conceptualizing the inference-reference process occurring during this ongoing attempt of recognition of appropriate forms is to consider meaning-making as unfolding in institutions (Douglas, 1987, 1996), which produce thought collective and thought styles (Fleck, 1934/1979). A thought style can be viewed as a kind of shared semiosis, by which people infer similar meanings from signs perceived in a same way, in a common recognition of forms. This common recognition of forms can be seen as a *seeing-as* (Wittgestein, 1953/1997), which is a habit of perception, and make possible the *joint-inferences*. The whole teaching-learning process can be viewed under this description (Sensevy, Tiberghien, Santini, Laubé & Griggs, 2008).

1.3 The logic of practice: the grammar of situations

In analyzing the social world, our concern is a *grammatical* one. We do think that every practice is unfolded according a specific logic, which over-determinate a great deal of it. Thus, as researchers we take a *grammatical stance*, which means that we try to understand the specific situational logic, the *peculiar grammar*, of a given practice. This concern logically stems from the conception of cognition and language we outlined below. If meaning-making is a matter of recognition of forms which are given by the collectives we are in, the description of meaning-making process rests on the identification of such forms, that is, a grammatical perspective. We must point out that a general way of understanding the logic of the practice lies in the comprehension of the situations in which this very concrete practice unfolds. The logic of practice is the logic embedded in the situations of practice. This kind of description helps understand why the meaning making process is viewed as mainly analogical. If the logic of practice is determined by the logic of the situations of the practice, meaning is made by relating the actual situation in which we are acting to the previous ones which resemble to the current one.

1.4 Game, situation, institution

In order to describe the grammar of the situations, we use a way of describing the social world in terms of *games*, by developing a "bourdieusian" perspective (Bourdieu, 1992). We consider the human activity as developing in games. By using the notion of game, we may use the following descriptors: the stakes of the game; the investment of the players in the game; the "feel for the game" that the players can or cannot display; the different kind of *capitals* related to the different games, that is, a way to acknowledge power phenomena in the social world. Thus the game is for us a fundamental grammatical structure, as a model of the social world, and also as a mean to relate institution and situation. Learning to act in a specific part of the social world is learning to play a certain game in situations embedded in institutions.

2. THE JOINT ACTION THEORY IN DIDACTICS: SOME TOOLS

2.1 The Didactic Game as a general pattern

We can try to describe the didactic interactions between the teacher and the students as a game of a particular kind, a *didactic* game. What are the prominent features of this game? It involves two players, A and B. B wins if and only if A wins, but B must not give directly the winning strategy to A. B is the teacher (the teaching pole). A is the student (The studying pole). This description allows us to understand that the didactic game is a collaborative game, a *joint* game, within a joint action (Clark, 1996). If we look at a didactic game more carefully, we see that B (the teacher), in order to win, has to lead A (the student) to a certain point, a particular “state of knowledge” which enables the student to play the “right moves” in the game, which can ensure the teacher that the student has built the right knowledge. At the core of this process, there is a fundamental condition: in order to be sure that A (The student) has really won, B (The teacher) must remain tacit on the main knowledge at stake. The teacher has to be *reticent* in order to let the student build proper knowledge, *her* proper knowledge. The teacher has to withhold information, because the student must act *proprio motu*. The teacher’s scaffolding must not allow the student to produce the “good behavior” without mastering the adequate knowledge. This *proprio motu* clause is necessarily related to the *reticence* of the teacher. Indeed, according to us, the didactic game, with the *proprio motu* clause and the teacher’s reticence, provides a general pattern of didactic interactions.

2.2 From the Didactic Game to the Learning Games

The Didactic Game refers to what we consider to be the fundamental grammar of the teaching-learning process. In order to deeply characterize this process, we use a system of concepts that we aim to unify under the notion of Learning Game. Learning Game, as a way of describing the Didactic Game as it occurs *in situ,* requires itself a structure of particular descriptors : the didactic contract/milieu doublet ; the genesis triplet (mesogenesis ; chronogenesis ; topogenesis) ; the game quadruplet (defining, devolving, monitoring and managing the certainty/uncertainty dialectic, institutionalizing). In the following, we will give some rapid descriptions of this system of concepts. First of all, a Learning Game can be identified by describing the didactic contract and the milieu referring to the piece of knowledge at stake.

The didactic contract and the milieu

We consider the didactic contract (Brousseau, 1997) according to a threefold viewpoint. The didactic contract can be viewed as an implicit system of mutual *expectations* (Mauss, 1989) between the teacher and the students, about the knowledge at stake, an implicit system of *joint habits* (Dewey, 1922) about this knowledge, and an implicit system of mutual attribution of *intentions* (Baxandhall, 1985). It is important to point out that this definition emphasizes the permanent features of the contract, and may explain the analogical process of meaning-making. We consider the didactic milieu under a 2 components description. On the one hand the milieu is a cognitive context, as a common ground, which notably provides the expectations and the mutual attributions of intentions on which the didactic contract rely. With this respect, the milieu is a system of shared meanings which makes possible the joint action. But this kind of description is not efficient enough to provide a good understanding of the teaching-learning process. One has to acknowledge that in order to learn, the students have to encounter an antagonist milieu (that Brousseau called adidactic milieu), a kind of *resistance* to their action, which is also a resistance to the joint action. Thus this notion refers to the part of knowledge that the students cannot directly assimilate, which resists to their habits, and which prevents them to play the *right game*. The way in which the milieu provides such a resistance can be figured out (or not) *a priori* by the teacher, and even modelled by a researcher. It is important noticing that encountering the resistance of the milieu requires a certain grasp of consciousness. Indeed, by experiencing this resistance, the students have to encounter their ignorance, and the need for a specific piece of knowledge which will bridge this "ignorance gap".

The dialectic between contract and milieu

When students try to play a learning game, some moves are directly given to them by the habits of action related to the knowledge they have recognized as the knowledge at stake. Some of these moves don't enable them to act accurately to meet the didactic situation requirements. In some cases, it is why they encounter a resistance to their action, and they just no longer play the game. It is critical to understand that these encounters and the shared awareness of their reality are a matter of joint action. Among all categories which are used for the description of learning games, the relationship between contract and milieu holds a prominent position. In order to characterize the didactic joint action, one has to identify how the students orient themselves, either by enacting the didactic contract habits or by establishing epistemic relations with the milieu. It means that empirical studies have to reveal what kind of dialectic is built between the "contract-driven students' orientations" and "the milieu-driven students' orientations", in order to understand the Didactic Joint Action and the way mathematical knowledge is processed.

The game quadruplet

What we call "the game quadruplet" is a set of categories that we use to describe the way the teacher has the students playing the game in the joint action (Sensevy, Mercier, Schubauer-Leoni, Ligozat, & Perrot, 2005). *Defining.* The defining process can be viewed as a way of introducing the definitory rules of the learning game, in order for the students to be able to play this game. *Devolving.* When a game is defined, it has to be accepted by the students. That means that the students have to elaborate an adequate relation to the milieu. *Monitoring, managing the certainty/uncertainty dialectics.* The monitoring process refers to any teacher's behaviors produced to modify the students' behavior in order to enable them to produce the relevant strategies they need to win the game. In doing so, the teacher plays on the level of certainty/uncertainty of the students' action. *Institutionalizing*[1]. In the ongoing didactic process, the teacher needs to recognize parts of the targeted knowledge in the students' activity as the relevant one for the learning game at play. In doing so, it

[1] The terms 'devolving' and 'institutionalizing' refer to Brousseau's concepts (1997).

makes the student understand that their activity reached the knowledge at stake, which is not only the "classroom knowledge", but also the knowledge of a social community, which is larger than the school community.

At another scale and with other purposes, we consider a triple dimension that describes the teacher's work, relative to starting and maintaining a didactic relationship (Chevallard, 1991, 1992; Sensevy, Mercier, Schubauer-Leoni, Ligozat, & Perrot, 2005) in the playing of the game.

The genesis triplet

Mesogenesis (i.e. the genesis of the milieu) describes the process by which the teacher organizes a milieu, with which the students are intended to interact in order to learn. *Chronogenesis* (i.e. the genesis of the didactic time) describes the evolution of knowledge proposed by the teacher and studied by the students, as it unfolds in the joint action. The teacher has to monitor the knowledge process through a lesson or several lessons, in order to meet his didactic intentions. *Topogenesis* (i.e. the genesis of the positions) describes the process of the division of the activity between the teacher and the students, according to their potentialities. The teacher should define and occupy a position, and enable the students to occupy their positions in the didactic process.

3. AN EMPIRICAL ILLUSTRATION

We focus now on an empirical example. The learning game occurred in an adidactic situation: the puzzle situation (Brousseau, 1997, p. 177) within a very large "didactic engineering" (N & G. Brousseau, 1987). I will make a first analysis of this episode, before trying a more general description of the same episode. The puzzle situation is a first situation for the study of linear mappings. It is put to students as following (Brousseau, 1997): "Here are some puzzles (Example: "tangram"). You are going to make some similar ones, larger than the models, according to the following rule: the segment that measures 4 cm on the model will measure 7 cm on your reproduction. I shall give a puzzle to each group of four or five students, but every student will do at least one piece or a group of two will two. When you have finished, you must be able to reconstruct figures that are exactly the same as the model". *Development*: after a brief planning phase in each group, the students separate. The teacher has put an enlarged representation of the complete puzzle on the chalkboard.

In the studied episode, as usual in this case, the students have added 3 cm to every dimension. The result, obviously, is that the pieces are not compatible. The teacher comes to a group at this moment. We give the transcription of the dialogue between the teacher and the students.

The puzzle episode

1.	Student	There's a problem it looks as if one is missing
2.	Teacher	There's a problem, yes
3.	Student	But already here it's leaning a lot here and then it's there
4.	Teacher	Yes and it should be leaning in the same way?
5.	Student	Here we can see that the pike/point it touches the other one here again there is a problem and here it should be there it does like this there it does like this it would have been correct
6.	Teacher	And everywhere here you have added 3 are you sure you've added 3
7.	Student	yes
8.	Teacher	1,2,3, 1,2,3, 1,2,3
9.	Student	Well not to this one
10.	Teacher	1,2,3, have you added 3 everywhere?
11.	Student	Well it is correct

12.	Teacher	Then what must be challenged?
13.	Student	Well It's wrong well this piece is a good one
14.	Teacher	Well no it's not because it doesn't make up the good puzzle
15.	Student	Here it doesn't make 3
16.	Teacher	Where 3?
17.	Student	It only makes 2
18.	Teacher	Well 3? It's 3 more where?
19.	Student	On each side
20.	Teacher	If I were you I'd think about the method I used maybe this is what's not good
21.	Students	Yes
22.	Teacher	Maybe it's you're sure you've added 3 you didn't make any mistakes when you cut out the pieces, ok ? Everyone has cut on the lines?
23.	Students	Yes
24.	Teacher	Well so maybe you mustn't add 3 you must do something else
25.	Tony	But from 4 to find 7
26.	Teacher	Ah
27.	Student	There's a problem here too
28.	Teacher	Are you listening Tony
29.	Students	Yes
30.	Teacher	Go on try to look into this problem

3.1 The puzzle episode: a first description

A possible structure of the episode

In ST (Speech Turn) 1, the student acknowledges that "there is a problem". We can analyze the excerpt by structuring it into for parts : in the first part, from 1 to 11, the teacher want the children to agree that if there is a mistake, it is not a measurement mistake; the ST 12 (*Then what must be challenged?*) is the teacher's first try to give to the students an incentive to challenge their method, but without effect; in the second part, from 13 to 19, the teacher and the students return to the discussion of the measurement method, notably by arguing about what is a "good piece" (13-14); in the third part, from 20 to 26, the teacher takes a high topogenetic position, in order to focus the students' attention on the "proper signs" of the situation; in the forth (last) part, one can think that the students are beginning to challenge their methods (25-27), so the teacher leaves the students and goes to another group.

Some teacher's moves in the Joint Action

We can focus on several teacher's moves in this excerpt. 1) In ST 4 (*It should be leaning in the same way?*), the teacher holds a "come-along position", which means a low position in the topogenesis, at the same level as the students. We can think that a good students' answer could be something like "Yes, because the model and the reproduction must have the same dimensions, the same properties" (this answer would be based on the conservation of proportions), but the students do not really understand the question. 2) In ST12 (*Then what must be challenged?*), the teacher's move is produced in order to make the students understand that they have to change their way of conceiving the problem. It is worth noticing that this calls for a different position from the teacher: not a "come along position", but an "analysis position", in which the teacher does not use the same kind of reticence about his knowledge. But this move does not work, for the students go on discussing about their measurement. 3) In ST 20, (*If I were you I'd think about the method I used maybe this is what's not good*) the teacher takes a higher position, in a very interesting utterance: "*If I were you*" functions as a prominent sign in the didactic contract. For the students, that may mean that the teacher is saying an important thing; by using the word "*method*" the teacher draws the students' attention to the fundamental meaning in this episode; 4) In ST 22, the teacher makes a summary of the students' work that one could paraphrase by saying "Are sure that your measurement was right ?". It functions as a kind of frame for an inference which could be: if you are sure that your measurement was right, then you have to challenge the method. 5) In ST 24 (*Well so maybe you mustn't add 3 you must do something else*), the teacher draws herself the inference (if it is not a measurement error, then it is a method error). Tony's reaction is very informative of his endorsing of the additive strategy; it's a kind of encounter of ignorance. For the first time in the episode, the additive strategy is questioned, which may function as a sign for the teacher that the learning process is going on.

3.2 The puzzle episode: a re-description

Here the learning game takes place inside an *adidactic* situation (Brousseau, 1997)[2]. First of all, the students have to encounter their ignorance, with the resistance of the milieu. In this learning game, as we have seen, they have to make a clear distinction between what is a *measurement error* and what is a *method (mathematical) error*. In order to move the didactic time forward, the teacher has to be sure that the students are convinced they have not made a measurement error. It is a necessary condition for them to challenge their method (i.e. the additive method). We can re-describe the episode using some theoretical tools of the JATD.

Reticence and proprio motu ; topogenesis and chronogenesis

The topogenetic characterization of this learning game enables us to understand how the teacher is progressively taken more and more responsibility in the didactic transactions. From a low topogenetic position (ST2, *there's a problem*, *yes*), he reaches a rather high topogenetic position (ST 24, *Well so maybe you mustn't add 3 you must do something else*). At the beginning of the episode, the reticence is very important, and the teacher does not unveil his didactic intentions. At the end of the episode, even if the teacher has displayed a part of his intentions, the reticence remains important. Indeed, nothing has been said about the proportional reasoning, which is at the core of this situation. The state of the milieu makes possible such an evolution, for there is a kind of agreement between the teacher and the students that the measurement is right. Thus we can acknowledge the specific interplay between chronogenesis and topogenesis in this rather short episode. The high topogenetic position is possible only because the didactic time - which is the knowledge time - has gone by, as we can see in the comparison of ST 2, 12, and 24. The teacher's "feel for the game" enables her to accomplish gradually this topogenetic rising while keeping an effective didactic reticence.

[2] In order to be understood properly, this episode would have to be replaced in a more general structure, investigated at different scale-levels. We are focusing here on the micro-level of the didactic transactions, but a complete inquiry would necessitate a meso-level and a macro-level investigation (on this point, see Ligozat, 2008). This is a fundamental methodological issue for the Joint Action Theory in Didactics, which rests on the necessity to provide enquiry processes with a plurality of description levels, using for this purpose specific tools (in particular *synoptic table* and *didactic plot*).

The contract-milieu dialectic

At the beginning of the episode, the students are caught in the didactic contract enacted by the situation. As a student said, “from 4 to find 7” one *has to make* an addition. This “additive contract” could be considered as a thought style in this episode, which provides a way of perceiving and a way of acting. Another feature of the didactic contract at play could be found in a lack of experimental culture which prevents the students to distinguish the “measurement realm” from the “conceptual realm”, and which brings a kind of “experimental fuzziness”. Thus the present learning game stems from the students’ observation that the puzzle pieces do not fit together. This observation has to be seen as a resistance of the milieu, a relevant feedback for the modification of the students’ strategy. But this resistance is not obvious for the students, and the teacher’s work consists of helping the students “read” the milieu. For the researcher (and for the teacher as well) a fundamental aspect of this episode consists in acknowledging how the contract/milieu dialectic needs to be built in the transactions. The milieu feedback is not at all naturally perceived by the students. In the uncertain didactic transactions, what counts as an evidence for the teacher (the pieces do not fit together), which provides an accurate inference (the additive strategy does not work) is very far from the students’ relationship to the milieu, given that this relationship is shaped by i) the “additive contract” and ii) the “experimental fuzziness”. The students have to build another relationship, and they can’t do that alone. The teacher’s monitoring is fundamental to foster the students’ relevant relationship to the milieu and its events, which will enable them to “resist” to the contract habits and to renew them. In that, for the teacher, enacting the contract-milieu dialectic in the didactic transactions is a way of taming the uncertainty while building a relevant certainty, and enabling the students to accurately recognize the “empirical facts”.

4. NETWORKING MATHEMATICS EDUCATION THEORIES: SOME BRIEF CONCLUSIVE REMARKS

0. The Joint Action Theory in Didactics (JATD) is a didactical Theory. It responds to the fundamental definition of Didactics as a science: the science of conditions and constraints under which *the diffusion of knowledge* is enacted. In order to situate this theory (JATD) in relation with the Theory of Didactic Situations and the Anthropological Theory of the Didactic, we can argue that while these two theories first focus, from a logical point of view, on the nature of *knowledge* (what is knowledge which is taught?), the JATD first logically focus on the *diffusion process* (What is going on when a specific piece of knowledge is taught). This is what we may call *the actional turn* of the JATD. This difference of logic means a difference of problems: the kind of problems the JATD attempts to solve, in a bottom up process, are that of the *didactic action*.

1. Prediger (2008) proposes an interesting way of characterizing theoretical conceptualizations according to three types, as idealized poles: “individual learning”, “class teaching”, “institutional structuring”. In this perspective, it seems to me interesting to notice that a crucial point for the JATD consists in an attempt to understand how the institutions, in Douglas’ meaning (1987, 1996) shape the individuals’ personal life in thought styles (Fleck, 1934/1979). So, one can say that in the JATD the “institutional

concern" is the first one. It does not mean that the JATD is not interested in "individual learning" or in "class teaching". On the contrary, we believe that the development of mathematics education theories needs a theory of didactic experience, if we call "didactic experience" these life events which enable people (and not only students or teachers) to gain knowledge as power of acting. But an essential feature of the JATD lies in the theoretical principle which assumes that meaning-making is mainly at work in the situations that institutions enact.

2. In the same paper, Prediger (2008) proposes another interesting way of characterizing studies with respect to the "prioritized types of research intentions". Thus the studies are located on an axis from "improved understanding" to "improved practices". As the other theoretical endeavors in French didactics, the JATD is rather on the "improved understanding" pole. But I would like to say that this type of reasoning could be dangerous, if researchers do not succeed in building a kind of normativity. This normativity, rationally and empirically grounded, could enable them to identify some principles in order to understand the didactic value of teaching-learning practices.

3. As a conclusion I would refer to Radford's paper (2008) about the problems of networking theories. In this paper, Radford considers theories as "flexible triples" of "principles, methodologies, and paradigmatic research questions" (Radford, 2008, p. 322). He then argues that "If we dig deep enough, we will find that difficult to connect theories are more likely to have fundamental differences in their system of principles" (Radford, 2008, p. 325). As any theory, the JATD rests on some principles. It seems to me that it could be useful to distinguish epistemological principles, which represent a theory of knowledge for a given theory, from theoretical tools, which are used directly in the enquiry process. In a good deal of published papers, the epistemological principles in the background of the research, which one can see as the roots of the theoretical tools, are not really worked out. It seems to me very important to clarify these epistemological roots if we want to network theories. In this perspective, a primary concern, following Kidron et al (2008), could be to shed more light on the role of social interactions in theoretical approaches, with respect to their epistemological roots. As Kidron et al show, all the researchers agree on the importance of taking into account this type of interactions in their theoretical frameworks, but what is the meaning and the value of such an agreement ?

References

Baxandall, M. (1985). *Patterns of Intention: On the Historical Explanation of Pictures*. New Haven: Yale University Press.

Bourdieu, P. (1992). *Language and Symbolic Power*. Cambridge: Polity Press

Brousseau, G. (1997). *The theory of didactic situations in mathematics.* Dordrecht: Kluwer.

Brousseau, N. & G. (1987). *Rationnels et décimaux dans la scolarité obligatoire*. Bordeaux: Ladist.

Chevallard, Y. (1985/1991) *La transposition didactique. Du savoir savant au savoir enseigné*. Grenoble: La Pensée Sauvage éditions.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en didactique des mathématiques, 12*(1), 73-112.

Clark, H. (1996). *Using language*. Cambridge: Cambridge University Press.

Dewey, J. (1922). *Human nature and conduct.* New York: Modern Library.

Douglas M. (1987). *How institutions think*. London: Routledge.

Douglas M. (1996). *Thought styles*. London: Sage Publications.

Fleck L. (1934/1979). *Genesis and development of a scientific fact*. Chicago: The University of Chicago Press.

Kidron, I., Lenfant, A., Bikner-Ahsbahs, A., Artigue, M., & Dreyfus, T. (2008). Towards networking three theoretical approaches: the case of social interactions. *Zentralblatt für Didaktik der Mathematik*, 40, 247-264.

Ligozat, F. (2008). *Un point de vue de didactique comparée sur la classe de mathématiques.* Thèse de doctorat en cotutelle, Université de Genève et Université de Provence.

Lorenz, K. (1994). Pragmatic and semeiotic: the percean version of ontology and epistemology, *In G. Debrock and M Hulswit (eds.). Living Doubt* (p. 103-108). Dordrecht : Kluwer Academic Publishers.

Mauss, M. (1989). *Oeuvres*. Paris: Minuit.

Prediger, S. (2008). How are theoretical approaches expressed in research practices? A report on an experience in comparing theoretical approaches with respect to the construction of research problems. *Zentralblatt für Didaktik der Mathematik*, 40, 277-286.

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *Zentralblatt für Didaktik der Mathematik*, 40, 317-327.

Schubauer-Leoni, M.-L., Leutenegger, L. & Forget, A. (2007). L'accès aux pratiques de fabrication traces scripturales convenues au commencement de la forme scolaire: interrogations théoriques et épistémologiques. *Education & Didactique*, *1* (2), 9-36.

Sensevy, G., Mercier, A., Schubauer-Leoni, M-L., Ligozat, F. & Perrot, G. (2005). An attempt to model the teacher's action in the mathematics class. *Educational Studies in Mathematics, 59* (1, 2, 3), 153-181.

Sensevy, G. & Mercier, A [eds] (2007) *Agir ensemble. L'action didactique conjointe du professeur et des élèves*. Presses universitaires de Rennes.

Sensevy, G., Tiberghien, A., Santini, J., Laubé, S. & Griggs, P. (2008). Modelling, an epistemological approach: cases studies and implications for science teaching. *Science Education*, *92*(3), 424-446.

Wittgenstein L. (1997). *Philosophical Investigations.* Oxford: Blackwell.

THE TRANSITION BETWEEN MATHEMATICS STUDIES AT SECONDARY AND TERTIARY LEVELS; INDIVIDUAL AND SOCIAL PERSPECTIVES

Erika Stadler

Växjö University

The aim of this paper is to illustrate how an empirical research interest in the transition between mathematics studies at secondary and tertiary levels generates a need for different theoretical approaches. From interviews with teacher students before and during their university studies in mathematics, three crucial aspects of the transition have been discerned; Mathematical learning objects, Mathematical resources *and* Students as active learners. *Whereas the two former have both individual and social dimensions, the latter can be regarded as relational, constituting a link between the learning environment and the student in his or her intention to learn mathematics.*

Keywords: teacher students, transition, individual, social, grounded theory

INTRODUCTION

My ongoing research project examines the transition between mathematics studies at secondary and tertiary levels, from now on termed "the transition". This research interest stems from novice university students experiences with increased difficulties and changes in the conditions of mathematics studies at university, compared to upper secondary school. When novice university students begin their studies at university, they learn mathematics in a new learning environment. From a student's perspective, this situation presents new challenges in terms of, or changes in, their knowledge, skills and self-image. Dynamic processes are going on, whereby students and their learning environment are mutually influencing each other. There are no obvious theories or methods at hand for dealing with this complex and extensive research area. Consequently, this study exemplifies the question raised by Arzarello, Bosch, Lenfant and Prediger of "how empirical studies contribute to the development and evolution of theories" (2007, p. 1620). Thus, an important part of the study has been to develop an analytical framework for the transition as seen from a student's perspective. In this paper, I will give an account of the theoretical considerations this empirical problem brings to the fore.

TRANSITION-RELATED RESEARCH

Learning mathematics at university level is a well examined area. Many studies have focused on students' learning and understanding of specific topics within university mathematics, for example limits of functions, derivatives, linear algebra and group theory (Dorier, 2000; Juter, 2006; Nardi, 2000). Other studies have considered how students struggle with advanced mathematical thinking, and with changes in the subject itself, including transformations from concrete and intuitive to more abstract, formal and general forms of mathematics (Tall, 1991) or demands for new ways of approaching the mathematical content (Lithner 2003, Schoenfeld, 1992). A common characteristic of these studies is an approach that focuses primarily on the students as individual learners. From a more situated and cultural perspective, the issue of transition between different contexts of mathematical practices has been carefully examined by de Abreu, Bishop and Presmeg (2002). They define transitions as individuals' experiences of movements between contexts of mathematical practices. The transition as seen from a student's perspective can be captured by studying students' actions and interactions in a learning situation, looking for traces of conflict between different learning cultures, or variations of meaning that students ascribe to phenomena in the learning situation.

Artigue, Batanero and Kent (2007) suggest that research in learning mathematics at the post-secondary level must go beyond notions of for instance advanced mathematical thinking and also involve more comprehensive perspectives on mathematical thinking and learning. In their article, they refer to Praslon, who states that the transition cannot be defined as a shift from school mathematics to formal mathematics, or from an intuitive approach to mathematics to a more rigorous one. Instead, the transition is rather a question of an accumulation of small changes in mathematical culture. It is a shift from studying specific mathematical objects towards an extraction of mathematical objects from more general conditions. It is a change from applying specific algorithms to a category of tasks towards general methods and techniques. According to Praslon this is a consequence of the increment of the mathematical content to be learnt, and the impossibility of learning a specific algorithm for every kind of task in a relatively short period of time.

By gathering many research studies from different areas with different perspectives, it is possible to grasp a more complete picture of the transition. This has been done in a recently published study by Gueudet (2008), who states that the transition involves individual, social and institutional phenomena that call for different theoretical approaches. From my brief overview of transition related research it can be concluded that research concerning the transition has been conducted both from individual (von Glasersfeld, 1995), situated (Wenger, 1998), and cultural perspectives (Säljö, 2000). To examine the transition from a student's perspective, where the transition is defined as learning in a new environment in light of previous experiences is to simultaneously consider individual and social perspective on learning. Thus, the challenge is to combine an individual a social perspective on a local level within one empirical study.

From a more general point of view, this issue refers to the discussion of whether individual and social perspectives on learning can be unified. Cobb and Yackel made an important contribution to this debate with their *Emergent perspective* (1996). Their notions of sociomathematical norms and mathematical beliefs and values coordinate an individual and a social perspective on the collaboration between the teacher and the students in classroom environments. The strong emphasis on interaction in the classroom can be regarded as a strength of this perspective. However, the transition from a student's perspective is not limited to the classroom. Instead, an essential part of the study must concern individual previous experiences of learning mathematics, requiring one to base findings on interviews. Thus, there is a mismatch between the methodological implications of Cobb and Yackel's Emergent perspective and the requirements of the research design of my study. My study requires a theoretical perspective that considers both an individual and a social perspective on the transition but from a methodological point of view, it requires more variety of data sources. Consequently, I was without a suitable theoretical framework and a pre-defined set of methods to follow to gather data and empirical considerations based on my definition of the transition had to serve as a starting point for the choice of research methods instead.

RE-ARRANGEMENT OF THE METHODOLOGICAL SEQUENCE

From a more general point of view this question also refers to a future challenge, raised during the Cerme 6 conference in Lyon, France, namely the discussion of how to find methodologies for networking theories, where the link between theory, empirical data and research results should be more highlighted. Methodological considerations link theoretical perspectives with appropriate research methods. Often, the formulation and intention of a research question is formulated within a theoretical discourse that results in a specific theoretical perspective. Thus, the research process, frequently used in mathematics education can schematically be described as follows:

Question → **T**heory → **M**ethod → **R**esult

Or alternatively:

Theory → **Q**uestion → **M**ethod → **R**esult

Here, theory may refers to a more comprehensive theoretical perspective, for example a social or situated perspective, but may also refer to a more local theoretical framework as the Emergent perspective. The point is that often decisions about method seem to follow almost automatically once the initial choices of research question and/or theoretical perspective have been described. My research approach has been somewhat different. The starting point for my study has been a real world situation, from which the aim and the definition of the transition were developed. Because the definition of the transition - the students' learning of mathematics in a new setting in the light of their previous experiences requires the study to combine an individual and a social theoretical perspective, there has not been a given choice of methodological approach. Instead, my intention to study the transition from a student's perspective has been used as a methodical starting point, whereby the results contribute to new theoretical approaches and relations.

This approach can be summarised as follows:

Aim → **M**ethod → **R**esult → **T**heory

With this rearrangement of the methodological sequence I want to emphasise how a real world problem implies a research process that ends up with a theoretical description of this phenomena. These descriptions have a local and specific character. However, based on their construction, they contain theoretical elements of both individual and social character. Thus, by studying them, conclusions can be drawn about how different theoretical perspectives come into play on a more general level. In accordance with my definition of the transition three main parts can be discerned, namely the students' previous experiences with mathematics studies, their learning of mathematics at university level, and the university as a new learning environment. To cover these parts empirically, I have collected different kinds of qualitative data from five teacher students during their first mathematics courses at university, i.e. individual interviews, observations from lectures and tutorials and written solutions to exercises and examinations. In this paper, I present some extracts from interviews with two of the students, Cindy and Roy. The pre-interviews were carried out after the students had enrolled at the university but before they had begun take courses in mathematics. The aim was to gain a picture of essential aspects of the students' understanding of mathematics studies in general and in particular of their experiences from upper secondary school. During their first courses in mathematics, the students were frequently interviewed to follow shifts in their thinking about mathematics and the learning of mathematics as they progressed through the courses. The interviews were audio-recorded and transcribed in full. Transcriptions have been analysed using methods inspired by Grounded theory (Charmaz, 2006). The data have been coded and sorted into categories, and axial coding has been used to analyse how the categories relate to each other. The result is a local theoretical description of essential aspects of the transition that could be discerned in the empirical data. However, these descriptions will contain aspects of individual and social theoretical perspectives from a more general point of view. How they interact within these concepts can also spread light of how different theoretical perspectives can be connected, coordinated, combined or networked.

RESULTS FROM INTERVIEWS

During the pre-interview, <u>Cindy</u> tells that she always liked mathematics and describes it as "her subject". She particularly enjoyed solving equations, which according to her demands accuracy and concentration. In lower secondary school, she was one of the best in her class, but in upper secondary school, she experienced that mathematics became more difficult. In her last courses, she had to "struggle to survive", and "integrals, strokes and such were not easy". A mathematics lesson usually started with a 10-15 minute lecture about the type of exercises the pupils were to work with. Next, the students would work individually with exercises from the textbook. During mathematics lessons, Cindy would collaborate with two classmates in a spontaneous group. By working together on the same exercise at the same time, they could explain to each other how to solve many exercises. To work on her own was meaningless to Cindy, because she would get stuck and could not continue on her own. When Cindy did not manage to understand the mathematical content, she simply tried to learn how to solve different types of exercises. She emphasises that there is a huge difference between knowing what to do and understanding mathematics, but her experience is that she often had to be content with the former. A new experience concerning exercises is that even if one finds the right answer, one cannot know if the solution is correct. For example, Cindy says that if she finds the limit of a function, she does not know if she has based her conclusion on the correct arguments or if she was simply lucky.

Cindy also thinks that another difference between mathematics studies at upper secondary school and university is that "it is harder" at university. She experiences that the mathematical content is more difficult and that everything is always completely new. During a mathematics lecture at the university an extensive amount of mathematics is covered, which results in many new things at the same time. This increases the risk of forgetting the first things that were said during the lecture. Cindy feels that the university teacher is good. When answering individual questions, he gives detailed explanations from the beginning. On the other hand, Cindy remarks that it is hard to get a straight answer or a simple explanation. Cindy feels that the most useful part of the lectures is when the teacher shows examples on the whiteboard, and when all steps in the solutions of the examples are demonstrated.

In the pre-interview, Roy tells that during upper secondary school he studied all available mathematics courses and got the highest grades. According to him, the first mathematics courses at upper secondary school were too easy. The majority of the mathematics consisted of using algorithms in a mechanical way and solving many similar exercises. This felt meaningless and bored him. It was not until later courses that Roy also met some challenges, which he defines as a need to "think for yourself". He tells that probability was one of his favourite subject areas, because it offered the opportunity to reason logically and to try different solution strategies.

Roy remembered that mathematics lessons usually began with a short demonstration by the teacher. During the remaining part of the lesson, the pupils worked individually or in spontaneous groups, solving exercises from the textbook. Roy's strategy was to look at the last exercises in the chapter. If he managed to solve them, he concluded that he could also solve the previous ones and that he had understood the content of the lesson. Most of the time, Roy worked on his own. However, if he did get stuck, he preferred discussing with his classmates instead of asking the teacher. He also frequently helped other students in his class and enjoyed explaining things to others. At university, Roy prefers working with peers rather than on his own, because it makes him more disciplined. From a social point of view, it is nice to meet with others and it makes studies more enjoyable. Often, he has solved more exercises than his peers have, but Roy likes to help the others solve exercises and feels it is a good opportunity to review the mathematical content.

When Roy compares mathematics studies at upper secondary school and university, he says that the main differences at university are longer lectures, a higher tempo, less time to work on exercises during lessons, the importance of “being in phase”, and really understanding. Another difference is that mathematics is no longer only a question of understanding or not understanding; it is also necessary to read about mathematics and learn some things by heart. This results in a need to study mathematics, not only to work on exercises. It is also essential to truly understand what one is doing and not just work on exercises. Roy says that he is very satisfied with the teacher, who works thoroughly on “building up the concepts with understanding” and states that he can “buy his explanations”. He also states that understanding is more important than ever, because if he is going to become a teacher, he needs a deep understanding to be able to explain even to gifted students. He feels very highly motivated.

ANALYSIS OF INTERVIEWS

From the interviews with Cindy and Roy, portraits of two individual students appear with very different experiences and abilities for mathematics studies. In the following, I will give an account of three central aspects that can be discerned from the interviews and that seem crucial to mathematics education in a learning environment, namely mathematical learning objects, mathematical resources and student as an active learner.

There are a number of objects and relationships that play an important role in students’ mathematics education, for example the teacher, peers, the textbook and time. Cindy’s and Roy’s stories illustrate how these come into play in different ways and how they support their learning of mathematics. Thus, empirical data implies that students use both tangible and intangible issues to accomplish what they consider as learning of mathematics. Results also show that to obtain mathematical learning demands making use of different entities in the environment. The *Mathematical learning object* refers to the main target of mathematics studies in a wider sense from the student’s point of view. This concept captures the very essence of what students think that mathematics is and what should be learnt. Though Cindy and Roy study the same mathematics courses, they give very divergent descriptions of the subject. While Cindy feels that mathematics gets harder and harder, Roy characterizes the increasing difficulty as a stimulating challenge. Cindy’s statement about integrals and strokes can almost be considered drivel, which in turn indicates a superficial view and memory of the mathematical learning object. Students use M*athematical resources* to obtain mathematical learning objects. In the interviews, Cindy and Roy explain how they collaborated with peers during mathematics lessons. However, while peers were an essential resource for Cindy to be able to solve exercises, peers rather had a motivational and self-confirmational function for Roy. Thus, a mathematical resource is relational rather than absolute and is constituted by students' usage of it. Which mathematical learning objects students focus and what they experience as understandable and meaningful can also be related to which mathematical resources the students are able to use. Different ways of interpreting mathematical understanding, their assignments and what it means to learn mathematics will also influence their mathematical study methods, which mathematical resources they choose to use, and how they view themselves as learners. One example that is worthwhile to examine further is their view of what a mathematical problem is, and what it means to solve it. Thus, it is plausible that how students perceive the mathematical learning object affects them as active learners, which in turn actualizes diverse mathematical resources and puts them into play in different ways.

There is a mutual relationship between mathematical resources and mathematical learning objects. Students use mathematical resources to obtain mathematical learning objects, but on the other hand, a mathematical learning object requires students' use of different mathematical resources. How they come into play depends on the characteristics of *Students as active learners*, which can also be discerned from the interviews. Students as active learners highlight the activities and actions they undertake to learn mathematics, and the intentions behind them. In the interviews, Cindy and Roy tell how they participated in the mathematics education and their thoughts and feelings about it. From these narratives, central aspects are, for example, the students' self-conception, motivation and identity. Cindy and Roy show clear differences between most of their learning activities, but they also carry out the same activity with different intentions.

As an example of how these three aspects interact, and how they interact in different ways for Cindy and Roy, I will return to an empirical example from the interviews. Even though Cindy wants to study mathematics, she often experiences the mathematical content as difficult. From her perspective, the content can be described as *inaccessible.* As a learner of mathematics, Cindy can be characterized as *dependent* with a view of the mathematical content as sometimes unmanageable and hidden. From her perspective, peers and teacher constitute a basic condition for her mathematical learning by helping her to find solutions to exercises and explaining things. By using them as a mathematical resource, she gains access to her mathematical learning object. For Roy, the mathematical content is *accessible*. To gain access is rather a question of his motivation for, and time spent on, studying. Roy can be described as an *independent* learner with a great portion of self-confidence in relation to the mathematical content. In his interaction with peers, they serve as a source of self-confirmation. Thus, peers as a mathematical resource have a more social and motivational character for Roy. The words *dependent* and *independent* as a description of students as learners and the *accessibility* or *inaccessibility* of mathematical content may be interpreted as inherited properties. However, this is not the way they should be understood. Instead, these characteristics are activated in the dynamic and interrelational interplay between the individual and the social environment. The concept of dependent-independent can rather be interpreted as an individual concept used with a social meaning. In the same way, the notion of access is used from an individual perspective. Thus, this example emphasizes and confirms that the transitions merge an individual and social perspective on learning. In relation to previous studies of mathematics studies at university level and the secondary-tertiary transition, it is obvious that the transition cannot be understood by limiting to learning a specific topic, ways of reasoning or advanced mathematical thinking. Instead, the interviews show that it is rather a question of an accumulation of small changes in the mathematical culture (Praslon in Artigue, Bataneri & Kent, 2007). However, these changes occur as a consequence of both changes in the learning environment and students' intentions and abilities to relate to them in a favourable way.

To further elucidate mathematical learning object, mathematical resources and the students as active learners, I will relate my analysis to the theoretical framework of Wenger (1998) regarding communities of practice. According to him, a practice is about meaning as an experience of situated activities. There are two interactively constituted processes involved in the negotiation of meaning within a practice, namely participation and reification. While the former is used in a common sense, the latter needs some clarification. According to Wenger, reification refers to "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'" (Wenger, 1988, p. 58). Thus, reification is tightly connected with the creation of meaning in relation to concrete or invisible objects and entities in the surroundings. From the above description, a parallel between Wenger's concepts of participation and reification on the one hand and my concepts of students as active learners and the mathematical content on the other can be discerned, whereby the mathematical resources constitute an interface between participation and reification or as the bridge between students as active learners and the mathematical learning object. Thus, it is clear that the concept of mathematical resources is more embracing than simply referring to something that gives rise to cognitive conflicts for the individual student from a constructivist point of view (von Glasersfeld, 1995). Neither does a mathematical resource equal a sociocultural artefact (Säljö, 2000). Instead, mathematical resources must be considered relational and dynamic. They come into play in the interaction between a student's intentional actions to learn mathematics in an actual situation, surrounded by a specific learning environment. From Cobb and Yackel's "Emergent perspective" (1996), students as active learners and the mathematical content can be related to both a social and a psychological perspective at all levels, while the mathematical resources appear between the individual and social columns in their model.

CONCLUDING REMARKS

My intentions with this paper is to show how a research interest can give rise to new theoretical concepts that do not fit in more established theoretical frameworks about thinking and learning. The case in question concerns secondary-tertiary transition.

The emergence of mathematical learning objects, mathematical resources and the students as active learners are a result of my initial statement that the transition is best understood from both an individual and a social perspective. For example, a mathematical learning object can be constituted by a specific mathematical concept or entity, but the shape of the learning object and which mathematical resources the student uses are both a matter of individual pre-knowledge, identity and overall aim with his or her studies, as well as the learning situation and availability of potential mathematical resources in the setting. There is a constantly ongoing interplay between these individual and social dimensions of the transition. The dynamical aspects of these categories capture essential aspects of the transition from the students' perspective. The transition may change the students' roles as active learners by contributing to shifts in their intentions with learning mathematics and in their actions in different learning situations. In turn these shifts may change the students' use of mathematical resources and their focus on different mathematical learning objects. This captures the core of the transition from the students' perspective, but also elucidates the interplay between individual and social theoretical aspects, raised from a complex "real world situation" that lacks an obvious choice of theoretical approach. The next step is to analyse observations of students working with mathematics in tutorials and in clinical settings, both when they work alone, under the guidance of the teacher and in collaboration with peers. These analyses are to contribute to a more sophisticated definition of the concepts, which can be used to characterize different learners and their paths through the transition.

REFERENCES

de Abreu, G., Bishop, A. J. & Presmeg, N. C. (2002). *Transitions Between Contexts of Mathematical Practices*. Dordrecht: Kluver Academic Publishers.

Artigue, M., Batanero, C. & Kent, P. (2007). Mathematics thinking and Learning at Post-Secondary Level. In: F. K. Lester, Jr. (ed.) *Second Handbook of Research on Mathematics Teaching and Learning: a project of the national council of teachers of mathematics*. Charlotte, NC: Information Age Publishing Inc.

Arzarello, F., Bosch, M., Lenfant, A. & Prediger, S. (2007). Different Theoretical Perspectives in Research From Teaching Problems to Research Problems. In: D. Pitta-Pantazi & G. Philippou (eds.) *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education*. 22-26 February, 2007, Cyprus.

Charmaz, K. (2006). *Constructing Grounded Theory. A Practical Guide Through Qualitative Analysis*. London: SAGE Publications Ltd.

Cobb, P. & Yackel, E. 1996). Constructivist, Emergent, and Sociocultural Perspectives in the Context of Developmental Research. *Educational Psychologist*, 31(3/4), 175-190.

Dorier, J.-L. (eds.) (2000). *On the Teaching of Linear Algebra*. Dordrecht: Kluwer Academic Publishers.

von Glasersfeld, E. (1995). *Radical Constructivism – A Way of Knowing and Learning*. London: RoutledgeFalmer.

Guedet, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, 67, 237-254.

Juter, K. (2006). *Limits of Functions. University Students' Concept Development*. Doctoral thesis, Luleå University of Technology.

Lave, J. (1988). *Cognition in practice: mind, mathematics and culture in everyday life*. Cambridge: Cambridge University Press.

Lithner, J. (2003). Students' Mathematical Reasoning in University Textbook Exercises. *Educational Studies in Mathematics*, 52, 29-55.

Nardi, E. (2000). *The Novice Mathematician's Encounter With Mathematical Abstraction: Tensions in Concept-Image Construction and Formalisation*. Unpublished Doctoral Thesis, University of Oxford.

Schoenfeld, A. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In D. Grouws (ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Säljö, R. (2000). *Lärande i praktiken – Ett sociokulturellt perspektiv*. Stockholm: Prisma.

Tall, D. (ed.) (1991). *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers.

Wenger, E. (1998). *Communities of Practice. Learning, Meaning, and Identity*. Cambridge: Cambridge University Press.

COMBINING AND COORDINATING THEORETICAL PERSPECTIVES IN MATHEMATICS EDUCATION RESEARCH

Tine Wedege

Malmö University, Malmö, Sweden

The author presents and discusses general issues related to combining and coordinating different theoretical perspectives and approaches in ongoing work on people's affective and social relationships with mathematics. The discussion is based on two concrete examples: Coordination of a sociological perspective (habitus) with an anthropological perspective (situated learning) in combination with a theoretical gender perspective on the analyses of qualitative data. The ambition of the paper is to bring a terminological clarification of differences between "perspective" and "approach" into the work on networking strategies for connecting theories.

INTRODUCTION

For the last 15 years a new international research field has been cultivated in the borderland between mathematics education and adult education. In order to study adults learning mathematics, conceptual frameworks and theoretical approaches has been imported from the two neighbouring fields and restructured (Wedege, 2001). Mathematics education research has welcomed and incorporated this new field where adult numeracy versus mathematical knowledge is continuously debated (FitzSimons et al., 2003). In this context, "diversity is not considered as a problem but as a rich resource for grasping complex realities" — as is stated in the call for papers from Working Group 9, *Different theoretical perspectives and approaches in research*, CERME6. As a consequence "we need strategies for connecting theories or research results obtained in different theoretical approaches", and Prediger, Bikner-Ahsbahs and Arzarello (2008) propose a terminology for dealing with this issue in the article "Networking strategies and methods for connecting theoretical approaches". As they state this is the "first steps towards a conceptual framework", which is based on the work in the Theory Working Group of CERME5:

The terminology of *strategies for connecting theoretical approaches* is presented as pairs of strategies (understanding others / making understandable; contrasting / comparing; combining / coordinating; synthesizing / integrating locally) within a scale of degree of integration from "ignoring other theories" to "unifying globally". The term *coordinating* is used when a conceptual framework is built by well fitting elements from different theories. This can only be done by theories with compatible cores. The term *combining* is used when theoretical approaches are only juxtaposed. This does not require complementarity or compatibility. Even theories based on conflicting principles can be combined. Finally, the term *networking strategies* is used to conceptualize those connecting strategies, which aim at reducing the number of unconnected theoretical approaches while respecting their specificity (Prediger et al., 2008, pp. 170-173). In this paper, I also follow Radford (2008) when he suggests considering theories in mathematics education as triples $\tau = (P, M, Q)$, where P is a system of basic principles "which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective" (p. 320); M is a methodology supported by P; and Q is a set of paradigmatic research questions.

The research project *Adults learning mathematics in school and everyday life* is an example of effort to grasp complex realities by connecting different theoretical approaches and perspectives (see http://www.mah.se/templates/Page____76536.aspx). Here, the purpose is to develop a comprehensive theory on conditions for adults learning mathematics, i.e. to establish an interdisciplinary theoretical framework to describe, analyse and understand the conditions of adults' learning processes — including social and affective aspects (Evans & Wedege, 2004; Wedege & Evans, 2006). In the research process, we find the relational interplay between theoretical investigations and empirical studies crucial when developing the theoretical framework, and different connecting strategies are used. Below, strategies of combining and coordinating are presented with two examples from this work. In the article "To know or not to know mathematics – that is a question of context" (Wedege, 1999), two theoretical perspectives (habitus and situated learning) are coordinated in the analysis of the data from a mathematics life history interview. In the paper "A gender perspective on adults' motivation to learn mathematics" (Wedege, 2008), a theoretical gender perspective was adopted in the analysis of existing qualitative data from a large English research project on adults' reasons for studying mathematics.

In this paper, I present and discuss theoretical and methodological issues from the work in progress on people's affective and social relationships with mathematics, drawing on the work of the CERME Working Group. The focus is on the influence of combining and coordinating different theories on the research process. But first, I shall propose a terminological distinction between a theoretical approach and a theoretical perspective.

THEORETICAL APPROACHES VERSUS THEORETICAL PERSPECTIVES

I adapt the understanding of "theory" as proposed by Prediger et al. (2008); i.e. the basic frame – or working definition – for discussion of conditions for connecting theories is "a dynamic concept of *theory* [or *theoretical approach*] whose notion is shaped by its core ideas, concepts and norms on the one hand and the practices of researchers – and mathematics educators in practice – on the other hand" (p. 176; my insertion and italic). According to this dynamic understanding, theories and theoretical approaches are constructions in a state of flux and theoretical approaches guide and are influenced by observation (p. 169). The notion of theory is broad when "theory" is synonymous with "theoretical approach". A first consequence is that theory is not only a guide for thinking but also for acting – for methodology. In the article

"Theories of mathematics education: Is plurality a problem?", Lerman (2006) examines the diversity of theories. He does not define "theory" but by looking at the examples and the proposed categorization of social theories within the mathematics education research community (1. Cultural psychology; 2. Ethnomathematics; 3. Sociology; 4. Discourse) it is obvious that Lerman's understanding of "theory" encompasses methodology and even problematique understood as a paradigm for mathematics education research (cf. Wedege, 2001). This conception is in contrast to Niss (2007) who presents a static definition of theory as a stable, coherent and consistent system of concepts and claims with certain properties; for example, the concepts are organized hierarchically and the claims are either basic hypotheses and axioms or statements derived from these axioms.

Another consequence of "theory" and "theoretical approach" being used as synonyms is that "theory" is implicitly distinguished from "theoretical framework", which does not automatically involve a methodology. The same goes for "theoretical approach" versus "theoretical perspective" and, in what follows, I shall suggest a terminological clarification of the latter pair.

I start by looking at the syntax and semantics of the two English nouns in the context of the debate in the Theory Working Group. According to the dictionary, "approach" is a verbal noun meaning the act of approaching (begin to tackle a task, a problem etc.). "Perspective" means a view on something from a specific point of view (seen through a filter) (Latin: perspicere = looking through). In our context, the noun does not have a verbal counterpart. The Danish verb "perspektivere" meaning "to put something into perspective" is not suitable here. In order to distinguish the two terms, I propose the following clarification: A *theoretical approach* is based on a system of basic theoretical principles combined with a methodology, as defined by Radford (2008), hence, guiding and directing thinking and action. A *theoretical perspective* is a filter for looking at the world based on theoretical principles, thus with consequences for the construction of the subject and problem field in research; that is the field to be investigated (cf. Wedege, 2001). For example, in the literature reference is often made to socio-cultural perspectives on mathematics education, simply meaning that social and cultural aspects of the educational phenomena are taken into account in research. Within the suggested terminology, it would not make any sense to talk about socio-cultural approaches without a reference to a specific theory, e.g. a socio-cultural approach – or problematique – like Engeström's (2001).

In order to exemplify how different theoretical perspectives which share an emphasis on the social dimension in mathematics teaching and learning lead to different interpretations and understanding of a short transcript of students' collaborative problem solving, Gellert (2008) compares and combines "two sociological perspectives" on mathematics classroom practice meaning. In order to "emphasise the theoretical grounds" of the two perspectives as he says, Gellert terms them "structuralist" and "interactionist" respectively. In this text, he is using the two terms "perspective" and "approach" alternatively without any terminological clarification. However, it seems that his choice of terms is deliberate and that his usage matches the distinction proposed above. He is talking about theoretical and methodological "approaches to research in mathematics education" (pp. 216, 220, 222) and "research approaches" (pp. 220, 221), and he concludes:

> The methodological approach I am sketching reflects a change of theoretical perspectives: Having identified relevant passages within the data material (from the structuralist point of view), these passages are analysed according to the standards of interactionist interpretation techniques (Gellert, 2008, p. 222).

In his discussion of the general issue of combining two theoretical perspectives, Gellert uses a piece of data – a short transcript of sixth-graders' collaborative problem solving. He states that "by selecting and focusing on this particular piece of data I have already taken a structuralist theoretical perspective" because, from this perspective, the passage is "a key incident of specification of inequality in the classroom" (p. 223).

COORDINATING AND COMBINING THEORETICAL PERSPECTIVES

A consequence of the terminological distinction between a theoretical approach and a theoretical perspective suggested above is this: In the network strategy of combining, theoretical approaches and theoretical perspectives are juxtaposed and they do not have to be complementary or compatible. But, in the strategy of coordinating, where well fitting elements from different theories are built into a conceptual framework, I consider only theoretical perspectives and they have to be complementary or compatible.

When theories are combined, a subject area is studied with different theoretical approaches. The area is structured into different problem fields to be investigated and different results are produced. When compatible or complementary theoretical perspectives are coordinated, the subject area is studied from an integrated perspective and one result is produced. According to Prediger et al. (2008) the strategies of coordinating and combining theories are mostly used for a networked understanding of an empirical phenomenon or a piece of data. In the following examples the aim of the networking is partly this and partly directed towards developing a theoretical framework.

Coordinating theoretical perspectives

As an example of coordinating theoretical perspectives for networked understanding of a piece of data, I have chosen the analysis of a life history interview (Wedege, 1999). In a narrative interview with a 75 year old woman, Ruth, about mathematics in her life there is a contradiction which is well known in adult education: many adults resist in learning mathematics in formal settings while they are mathematically competent in their everyday life. This particular woman, who had really bad experiences with mathematics in secondary school, went to a Technical School to be a draughtsman as 50 year old and she got the top grades in mathematics. But her dispositions towards having to do with mathematics did not change, neither did her beliefs about herself and mathematics. While some adults change their attitude to mathematics during a training course, others fail to do so. For some people, this means something for their image of themselves and their life project, for others not. These differences cannot be explained solely within the educational context and the students' current situations and perspectives. In order to expand the context for analysing learning processes and drawing a link to the lives lived by adult students, I have attempted to combine Lave and Wenger's concept of situated learning with Bourdieu's concept of *habitus,* i.e. systems of durable, transposable dispositions as principles of generating and structuring practices and representations (Bourdieu, 1980).

Lave and Wenger (1991) see learning as a social practice and the context of their analysis of learning processes is the current community of practice. The theory of *situated learning* is about learning as a goal-oriented process described as a sequence from legitimate peripheral participation to full participation. Throughout her life Ruth has participated in a number of different communities of practice (family, school, work, etc.). She learned a number of things in her mathematics lessons: that she was stupid at mathematics, that she was not interested in it, and that in any case mathematics had no relevance for her life. She was confirmed in this by never having failed in practical situations due to a lack of mathematics knowledge. When, much later in her life, Ruth got the highest grade in the subject of mathematics while being trained as a draughtsman, this did not change her idea of mathematics, the world around her, or herself. But the theory of situated learning does not present the possibility of explaining why her perception of herself had not changed, and why she never had any appreciation of mathematics.

Ruth's motivation to be a draughtsman made her overcome her blocks, but not her resistance to learning mathematics. Her intentions had changed but not her dispositions towards mathematics, incorporated through her lived life. According to the theory of Bourdieu, the habitus of a girl born 1922 in a provincial town as a saddler's daughter, of a pupil in a school where arithmetic and mathematics were two different subjects, at a time where it was "OK for a girl not to know mathematics", and the habitus of a wife and mother staying home with her two daughters is a basis of actions (and non-actions) and perceptions. Habitus undergoes transformations but durability is the main characteristic.

I have argued that the concept of habitus, developed and belonging in a sociological problematique as a concept of socialisation, can be coordinated[1] with Lave and Wenger's concept of situated learning in a problematique of mathematics education (Wedege, 1999). In the first place, Bourdieu emphasises that the theory of habitus is not 'a grand theory', but merely a theory of action or practice (Bourdieu, 1994). The

[1] The word I used in (Wedege, 1999) was "combined" and not "coordinated".

habitus theory has to do with why we act and think as we do. It does not answer the question of how the system of dispositions is created, and how habitus could be changed in a (pedagogical) practice. This means that the concept of habitus can be used in a descriptive analysis of the conditions for adults learning. Lave and Wenger's theory of situated learning is also a partial theory, a theory of learning as an integral part of social practice. They are precisely trying to find an answer to the question of how people's dispositions are created and changed through legitimate peripheral participation (Lave & Wenger, 1991). Bourdieu and Lave/Wenger both aim at challenging the dichotomies of subject-object and actor-structure. Both are critical of phenomenology and structuralism while simultaneously having social relations as the focus of their subject areas. Bourdieu set himself the task of constructing a theory of action as social practice and Lave a theory of learning as an integral part of social practice.

A common core – or basic principle – in both theories is the understanding of learning as social practice. Furthermore, the two theories reject the idea of internalisation of knowledge and attitudes/norms, respectively. They mention instead active incorporation. Thus, the theory of habitus, as a social practice theory, does not encompass the theory of situated learning, but I have argued that the two theories are compatible and that the concept of habitus, which is developed and belongs in a sociological problematique, can be imported into an educational problematique about adults' learning mathematics together with the concept of situated learning.

Combining these with a theoretical gender perspective

In the interview with Ruth, gender was an obvious aspect which might have been in the foreground of the analysis. The theories of habitus and of situated learning do not exclude gender aspects, but are a background dimension. In this section, I present another example of networked understanding of a piece of data – this time by combining the above with a theoretical gender perspective.

Complexity is a characteristic of the problem field in mathematics education, and diversity (gender, ethnicity, social class etc.) calls for multi- and inter-disciplinary studies and for different research methodologies. However, focus and methodology of any study are determined by its purpose, theory and research questions. For example Evans and Tsatsaroni (2008) have argued that research into gender within a social justice agenda requires both quantitative and qualitative methods.

When the research problem is formulated and the method and the sampling strategy are to be decided, the researcher has to choose among a series of factors and dimensions to reduce complexity. Gender is one of the aspects to be decided upon. In some studies, gender is a dimension in the *foreground*: the study is designed to investigate gender and mathematics – and gender is focussed in the purpose and the research question. In other studies, gender is a variable in the *background*: gender is just one independent variable among others.

Gender is in the foreground as an important analytical dimension in our on-going work on people's motivation and resistance to learn mathematics (Wedege & Evans, 2006). So far we have not designed a new empirical study with gender in the foreground but we have access to rich empirical data from 81 semi-structured interviews with students (2/3 female and 1/3 male) from an English research project on adult students' reasons for learning mathematics, "Making numeracy teaching meaningful to adult learners" (Swain et al., 2005). In this project gender is in the background: none of the research questions are about gender but information about gender is available in the data. In a pilot case study with one of these students, Monica, I have tried to adopt a gender perspective for a small part of this data (Wedege, 2008). The theoretical framework for this analysis consists of four analytical *gender viewpoints*[2] (*structural, symbolic, personal, and inter-actional*) (Bjerrum Nielsen, 2003). The analysis shows that the framework of gender viewpoints can be productive in locating gender in the data collected in the English project. The four gender viewpoints – separate or inter-connected – create new meanings to Monica's narrative.

From the structural gender viewpoint, gender constitutes a social structure, and men and women are, for instance, unevenly distributed in terms of education. For Monica, not having a high level of education has been a structural consequence of being a woman. As in many other families, girls were not educated in her family. They were brought up to fulfil traditional women's roles. Today, Monica is a single parent. In England – as in Scandinavia – the situation of being a single parent is closely connected with being a woman. Talking about reasons for attending the numeracy course, the students talked about the new governmental demands that single parents have to go back to work or alternatively go into training.

The core of our ongoing work is understanding motivation as a social phenomenon, which is also the case in the English project. Their theoretical framework is based on the work of, for example, sociologist Bourdieu and anthropologists like Lave (Swain, 2005 p. 31 ff) whom we have also used in our research. This theoretical choice had consequences for the questions asked to the students during the interviews, which in the case of Monica, for example, made it possible for her to talk about her childhood.

In the majority of studies in mathematics education, we find gender in the background. Hence, internationally, we have a large amount of data which has not been investigated from a theoretical gender perspective. In a recent overview of mathematics education research in Denmark and Norway, it was shown that very few studies were designed with gender in the foreground (Wedege, 2007). However, a series of Nordic researchers intend to bring gender into the foreground and, through the latest 15 years, they have presented papers with a focus on gender. These presentations were based on data from their own previous research (quantitative or qualitative stud-

[2] The term used by Bjerrum Nielsen (2003) is "perspectives". However, due to terminological constraints from the discussion in this paper, I have changed the term into "viewpoints".

ies) with gender in the background. That is, the researchers returned to their "own" data with questions related to their original problem.

CONCLUSION AND PERSPECTIVE

Diversity of theoretical approaches and perspectives is a challenge in research on adults learning mathematics, as in mathematics education research generally speaking. Inter-disciplinarity is also a significant feature of this field where theoretical frameworks are imported and restructured (Wedege, 2001). However, the researchers often import concepts from other disciplines, like psychology, sociology and anthropology, without any reflections on the process of import, integration and restructuration of the framework. Hence, there is a need for strategies for connecting theories from disciplines. Another problem is terminology and I see the present work, on developing terminology in parallel with strategies (Prediger et al., 2008), as very important in terms of quality. Hence, I hope that the proposed clarification of differences between the two terms "theoretical approach" and "theoretical perspective" will be adopted in the continuation of this work.

As mentioned above, the purpose – or the overall aim – of the research project "Adults learning mathematics in school and everyday life" is to develop theory, thus research with a *top-down profile* (cf. Arzarello et al., 2007). But if we look at the research process beginning in the 1990s, the aim of networking theories in the studies of adults learning mathematics alternates between top-down development and *bottom-up development* with the aim of understanding a concrete empirical phenomenon. The theoretical investigations and constructions iterate in continual interplay with empirical studies. In Wedege (1999), the aim of coordinating theories is understanding and explaining a concrete empirical phenomenon combined with intentions of theory development; in Evans & Wedege (2004) and Wedege & Evans (2006), the purpose is conceptual clarification and development; and in Wedege (2008), the intention is to combine with a theoretical gender perspective to revisit empirical data for new purposes. The aim of coordinating theoretical perspectives on habitus and on situated learning was to understand and explain a mathematical life history. But the arguments for compatibility of the two perspectives were general and not restricted to the data. In this and in the other studies, the development is driven by the concrete study combined with a general interest.

Combining and coordinating theories are steps on the road towards networking theoretical approaches in a new theory, but it is too early to say if our final networking strategy will be *synthesizing* between two or more equally stable theories or *integrating locally* some concepts or aspects of one theory into another more elaborated theory.

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REFERENCES

Arzarello, F.; Bosch, M.; Lenfant, A. & Prediger, S. (2007). Different theoretical perspectives in research for teaching problems to research problems. In D. Pitta-Pantazi & G. Phillipou (eds.), *Proceedings of the 5th congress of European society for research in mathematics education (CERME5)* (pp. 1618-1627). Cyprus: ERME.

Bjerrum Nielsen, H. (2003). *One of the boys? Doing gender in Scouting.* Génève: World Organization of the Scout Movement.

Bourdieu, P. (1980). *Le sens pratique*. Paris: Les éditions de minuit.

Bourdieu, P. (1994). *In other words*. Cambridge: Politic Press.

Engeström, Y. (2001). Expansive learning at work: Toward an activity-theoretical reconceptualization. *Journal of Education and Work,14*(1), 133-156.

Evans, J. & Wedege, T. (2004). Motivation and resistance to learning mathematics in a lifelong perspective. *Topic Study Group 6: Adult and lifelong mathematics education*. Available at http://www.icme-10.dk - Programme – Topic Study Groups.

Evans, J. & Tsataroni, A. (2008). Methodologies of research into gender and other social differences within a multi-faceted conception of social justice. *Adults Learning Mathematics – an International Journal, 3*(1), 13-31.

FitzSimons, G. E., O'Donoghue, J. & Coben, D. (2003). Lifelong Mathematics Education. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, F. K. S. Leung (eds.) *Second International Handbook of Mathematics Education* (pp.103-142), Dordrecht: Kluwer Academic Publishers.

Gellert, U. (2008). Validity and relevance: comparing and combining two sociological perspectives on mathematics classroom practice. *ZDM – The International Journal on Mathematics Education, 40*, 215-224.

Lave, J. & Wenger, E. (1991). *Situated learning. Legitimate peripheral participation*. New York: Cambridge University Press.

Lerman, S. (2006). Theories of mathematics education: is plurality a problem? *ZDM, 38*(1), 8-13.

Mertens, D. M. (2005). *Research and evaluation in education and psychology*. London: Sage.

Niss, M. (2007). The concept and role of theory in mathematics education. In C. Bergsten; B. Grevholm; H. Måsøval & F. Rønning (eds.), *Relating practice and research in mathematics education. Proceedings of NORMA 05. Fourth Nordic Con-*

ference on Mathematics Education (pp. 97-110). Trondheim: Tapir Academic Press.

Prediger, S.; Bikner-Ahsbahs, A. & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*, 165-178.

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM – The International Journal on Mathematics Education, 40*, 317-327.

Swain, J.; Baker, E.; Holder, D.; Newmarch, B. & Coben, D. (2005). *Beyond the daily application: making numeracy teaching meaningful to adult learners*. London: National Research and Development Centre for adult literacy and numeracy (NRDC).

Wedege, T. (1999). To know - or not to know - mathematics, that is a question of context. *Educational Studies in Mathematics, 39*(1-3), 205-227.

Wedege, T. (2001). Epistemological questions about research and practice in ALM. In K. Safford & M. Schmitt (Eds.), *Conversation between researchers and practitioners, The 7th International Conference on Adults Learning Mathematics (ALM7)* (pp. 47-56). Medford, Massachusetts, USA: Tufts University.

Wedege, T. & Evans, J. (2006). Adults' resistance to learn in school versus adults' competences in work: the case of mathematics. *Adults Learning Mathematics: an International Journal, 1*(2), 28-43.

Wedege, T. (2007). Gender perspectives in mathematics education: intentions of research in Denmark and Norway. *ZDM – The International Journal on Mathematics Education, 39*, 251-260.

Wedege, Tine (2008). A gender perspective on adults' motivation (and resistance) to learn mathematics. In J. O'Donoghue (ed.), *The changing face of adults mathematics education: learning from the past, planning for the future. Proceedings of the 14th International conference of Adults Learning Mathematics (ALM)* (pp. 289-298). Limerick: CAMET, University of Limerick.

COMPARING THEORETICAL FRAMEWORKS IN DIDACTICS OF MATHEMATICS: THE GOA-MODEL

Carl Winsløw

Department of Science Education, University of Copenhagen, Denmark

In this paper we propose a meta-model for comparing different theoretical frameworks in didactics, focusing on three components of the study object of didactics: a set of human beings with relations (e.g. students and teachers in a classroom), an organisation of human practice and knowledge, and a set of artefacts used to mediate and relate the previous two. We argue theoretically and through an example (related to the transition from secondary to tertiary education) that this meta-model helps identifying complementarities, similarities and differences among four leading theories or models of the didactical field, and thereby to facilitate rational justifications for selecting a theoretical framework with respect to a given purpose of research.

1. INTRODUCTION

The comparative study of theoretical frameworks in didactics of mathematics (for short, *didactics*) was the subject of a special issue of ZDM (no. 40, 2008), drawing on papers and discussions from working groups at CERME-4 and CERME-5 (cf. ermeweb.free.fr), as well as on other papers, many in previous issues of ZDM. Prediger et al. (2008, Fig. 1) subsumes the "landscape of strategies for connecting theoretical approaches" as ranging from "ignoring other theories" to "unifying globally", between which we find intermediate positions for "finding connections as far as possible (but not further)" that the authors call "networking strategies". Some consensus seems to have emerged to pursue the latter type of strategies, while considering the uses of a small number of theories (mostly 2-4) in concrete "cases" for research, such as studying or developing a classroom design based on a simple task. A general "metalanguage" to compare theoretical frameworks was proposed by Radford (2008, 320): a theory is considered as based on a triple consisting of a set of implicit and explicit principles of the theory, a methodology and a set of paradigmatic research questions. This idea seems to be applicable to theories in any field of research, and focuses essentially on aspects of the *epistemology* afforded by theories.

This paper proposes another, possibly complementary, approach to the issue: namely to compare the characteristic ways in which different theories build models of *the object of study in didactics*. The basic hypothesis is that significant differences among theories of didactics come from *focusing on different phenomena* within the complex reality of mathematics teaching and learning. In short, we propose a meta-model for the *ontology* of the theories, understood as the models they propose of their object.

2. EPISTEMIC SYSTEMS – THE GOA MODEL

Every science is about "something" – the *objects of study*. For an empirical science like didactics, which sets out to study a certain realm of mental, social or physical entities, the objects of study are delimited and to a certain extent constituted by the development of theoretical *models*. Such models are more or less *systemic* in the sense that they imply relations among the objects; models are not simply lists of independently defined objects.

Without assuming (or saying) much, the "object" of didactics can be loosely described as the teaching and learning of a specific knowledge domain. Teaching and learning implies subjects who teach and learn – that is, teachers and students, or more generally a *structured group of people* (where structure implies that members of the group may have different roles and relations to each other, such as being teachers or students). The knowledge domain itself can be modelled and analysed as a coherent *organisation of knowledge and practice*. Finally, knowledge and "knowers" (be they teachers or learners) cannot be related without *artefacts* of different forms (texts, media, other tools and materials of various sorts). Given these basic observations we suggest that the systems of objects studied in didactics can be described as a triple

$$(G, O, A)$$

where: G is a group of people structured by a certain set of relationships, O is an organisation of knowledge and practice which G enacts, and A is a set of artefacts which G uses to access and communicate in and about O. Notice how relations on $G \cup O \cup A$ are crucial not just to study but also to define the triple. We call such a triple an *epistemic system* (ES) because the system involves use, circulation, development or even production of knowledge. Of course, not all ES are likely to be objects of didactical research, but surprisingly many types could need to be taken into account.

An ES may be considered in synchronic and diachronic ways, corresponding to a snapshot of its state at a given time (or a shorter period where it can be considered as relatively stable), and to its development over a period of time. It is also important to notice that (G, O, A) may be considered as general systems corresponding to an *institution* (e.g. a professional community or workplace) where the artefacts may include such diverse objects as buildings, tools, texts and so on, giving identity and delimitation to the institution. Finally, an ES may be naturally divided into "subsystems" (G_i, O_i, A_i), such as different divisions within a workplace.

Here are four special cases which are of particular importance in didactics, in themselves and in interaction; they also show how varied phenomena ES include:

2.1. Didactic systems may be described as the case where G consists of one or (rarer) several *teachers* and a class of students, engaged in the teaching and learning of a knowledge organisation O while mobilising, possibly in different and changing ways, a set of artefacts A (including objects within the classroom). The knowledge organisation could be based on one or more problems or questions, mediated and tackled using A, and potentially mobilising or enabling the construction of the "intended knowledge or practice" (also part of O). In fact, these intentions – of the teacher(s) – are an important factor in didactic systems, but it could take many forms.

2.2. School systems consist of a certain collection of didactic systems, e.g. with *G* comprising all students and teachers of a given school, or of all schools within a given region or country; the boundaries of a school system (as regards all three components) are sometimes institutional boundaries in the sense that they are defined quite explicitly, such as by law, or they could be considered pragmatically as (observable) systems of persons common aims, practices, and material surroundings.

2.3. Teaching systems are parts of school systems but with *G* being a group of teachers, who may work alone, or together, to construct or reflect upon one or more didactic systems. The knowledge organisations and artefacts involved in such systems may, of course, also be quite different from those involved in didactical or adidactical systems. For instance, teachers could be involved in developing or sharing teaching plans and other teaching material (artefacts) related to the teachers' knowledge and practice enacted within didactic systems.

2.4. Noospheric systems consisting of a group of people *G* involved in generating, delimiting or defining all or parts of the knowledge and practice organisations *O* to be worked on in didactic systems, using or producing artefacts to this end; for instance, *G* may be one or more authors of a textbook (part of *A*) aimed to support *O*, or a group responsible for producing standards for school systems (*A* in this case involves documents setting up requirements or recommendations regarding the practice, target knowledge and artefacts of these). The term *noosphere*, originally coined by Chevallard (1985), ironically refers to the "thinking about" school systems which takes place outside these systems from a peripheric yet superior position.

3. The GOA model as a "meta-model" for comparing theories in didactics

The above model of ES can be thought of as a *meta-model* since its use in practice requires finer models for each of its components and their interrelations. Supplying these details, we recover several "real" models or theoretical frameworks commonly used in didactical research. We now do this for some important ones, familiar to us.

3.1. The theory of didactical situations (TDS, cf. Brousseau, 1997) considers, as its primary objects, *didactical situations* evolving around *didactical milieus* and regulated by *didactical contracts.* The situations are themselves modelled as the interplay between students and teachers (forming *G*) and the milieu, which in turn is a compound of both material elements (forming *A*) and a particular organisation O_M of practice and knowledge. The system as a whole is analysed in terms of a wider organisation *O* of intended and prescribed forms of practice and knowledge, which includes also a dialectic between *personal knowledge* of the different members of *G*, and *shared knowledge* which develops over time. This means that the entire didactic system (*G*, *O*, *A*) is considered diachronically, albeit mostly over shorter periods (corresponding to a lesson or a sequence of lessons). The didactical contract consists of (mainly) implicit rules which govern the whole system, in particular the interactions within *G* and between *G* and the milieu. In sum, this theoretical framework models *G* as consisting of a teacher and a group of students, with different relations to both *O* and *A*, a relation which varies over time and is interpreted as being governed of a rule system (contract) corresponding to expectations and obligations of the members of *G*. It can be said to be more "naturalistic" as regards *G* and *A* as such, and focuses particularly on the evolution of the relation between *G* and *O*. Moreover, diachronically, the theory focuses on subsystems existing at times where the teacher does not interact with the students, called *adidactical situations*; this refers to shorter time spans for a didactic system, which at other times involves interaction between teachers and students.

3.2 The anthropological theory of didactics (ATD) involves highly intricate models of *O* (mathematical and didactical organisations, cf. Chevallard, 2002), corresponding to forms of practice and knowledge related to mathematics and the teaching of mathematics, respectively. More precisely, it models both of these as organisations of *praxeologies*, each of which consist by definition in a quadruple (type of task, technique, technology, theory). Praxeologies are organised at various levels according to the techniques, technologies or theories they share. The researcher constructs a *reference model* to observe and analyse these organisations within different systems. Among artefacts explicitly considered in this theory are *ostensives* mediating and embodying the techniques and technologies of *O*, including also discursive media and tools. In this theory, *G* is mostly implicit, except for the strong emphasis on *institutions*, viewed as the human ecologies in which praxeological organisations live and between which they are transposed. The theory also contains a structured view of institutions successively determining each other at different levels (Chevallard, 2002), from a didactic system considered synchronically (e.g., Barbé et al., 2005), to the noospheric systems (including the level of societies) considered in diachronic development (e.g. Chevallard, 2002). Finally, a recent development in this theory, to describe the long term developments of didactic systems, is Chevallard's notion of *research and study programme* (see eg. Barquero et al, 2006), focusing again on *O* but with a community of learners *G* being perhaps more explicit in recent empirical studies of how such a programme evolves (ibid.). However, even more than TDS, the ATD focuses primarily on the analysis of the *O* component.

3.3 Socio-constructivist theory of mathematics learning (SCT) exists in many forms and variants; we consider here the approach to didactic systems developed by Cobb and associates (e.g. Cobb et al., 2001). As the name suggests, the model has dual roots (ibid., 119-120): on the one hand, in *constructivist* learning theories going back to pioneers such as Steffe, Skemp, and ultimately Piaget; and in *socio-cultural* theories, with a lineage involving names such as Bauersfeld, Lave and Vygotsky. The idea is to study the learning – in particular mathematics learning – of participants in a classroom situation (students, teachers and even researchers) both as individuals and for the group collectively within a socio-cultural context. In fact,

> there is an extremely strong relation between what we have described as the social and psychological perspectives that does not merely mean that the two perspectives are interdependent. Instead, it implies that neither perspective exists without the other in that each perspective constitutes the background against which mathematical activity is interpreted from the other perspective (ibid., 122).

The researchers' interpretation of classroom activity aims to clarify this dynamics of (mainly students') *individual beliefs* and *sociomathematical norms* developed and shared by *G* collectively. It is based on careful analysis of video recordings of classroom activity (as a primary form of data, among others such as field notes and interviews). This allows for observing not only *discursive* and *embodied* practices related to a mathematical task, and thereby the emerging organisation *O* of practice and knowledge found in the classroom, but also the role played by artefacts (discursive, semiotic, material…). It is important to note that while *G* and *A* are theorised e.g. as *communities of practice* and *semiotic ecologies* (ibid., 153), the individual and shared knowledge organisations (including beliefs and norms) are considered to *emerge* from the interaction within *G* and between *G* and *A*: *we take the local classroom community rather than the discipline as our point of reference* (ibid., 120). In other versions of SCT, such as Ernest (1997), a wider perspective is adopted.

3.4 The cognitive-semiotic theory (CST, cf. Duval, 1995) focuses on the relationships (mental schemes or processes) which exist for the members of *G* between a collection of signifiers (primary elements of *A*, organised in semiotic systems) and signifieds (mathematical objects within *O*). The fact that these relationships may be different for different members of *G* (and develop over time) is explicated in variants of this model through a *triadic* model of the sign relationship, including also the different interpretations or schemes for the relationship between semiotic artefacts and their "meaning". Particularly important for the objects of mathematics is *multimodal representations*, which occur in two distinctive forms (cf. Duval, 2000): different representations of an object within the same semiotic system (*register*), which are obtained by *treatments*, sometimes based on complicated algorithms; and representations in different *registers* (like a function being represented symbolically and graphically), obtained from each other by *conversion*. Coordination of different representations of a given mathematical object is a key requirement in many common mathematical tasks. Notice that this model may be applied to all kinds of ES, but with a special focus on "semiotic" artefacts and the corresponding schemes, and sometimes relatively implicit models of *O* (although for the case of mathematics, the mathematical objects and their properties are often considered as *constructed* or even *consisting* in those schemes, cf. Winsløw, 2004).

3.5 Comparison. The above four "snapshots" of theoretical frameworks enables a first comparison of them, as the modelling of certain *parts* of (*G*, *O*, *A*) occupy the foreground within each of them. In TDS, the interaction of teacher and students (*G*) around the didactical milieu (part of (*O*, *A*)), in ATD the praxeological organisations

(O) in their institutional context (G, A), in SCT the community of practice (G) with its evolving shared norms and individual beliefs which contribute to determine O, and in CST the semiosis and associated schemes (part of (G, A)) as a condition for accessing and enacting O. To compare these theoretical frameworks, it is crucial to realise that they model, to some extent, *different parts* of a common reality (such as a didactic system). To a much lesser extent do we find apparent oppositions in their basic constitution, such as the deliberate absence in SCT of reference models for O "outside the classroom", versus the strong emphasis on such models within ATD.

4. CASE: THE TRANSITION FROM SECONDARY TO TERTIARY

For about a decade, I have been studying the transitions problems which arise for students at the beginning of university programmes in mathematics, along with development projects aiming at enhancing the outcome of students' work. The difficulties students encounter – and the strategies one may envisage to help them overcome those difficulties – may be approached using any of the theoretical frameworks considered in the previous section (as well as others, of course). In this section, a simple example will be used to show how contributions associated to each framework are different because they model the relevant ES with different foci and notions. Notice that Gueudet (2008) presents an overview of literature explicitly addressing the transition from secondary to tertiary, including more theoretical perspectives than those considered here.

Globally, transition concerns *students* who move from one type of ES, (G, O, A), to another one, (G', O', A'), in which there may be some overlap in all three components, including (by definition) the students themselves within $G \cap G'$. An obvious place to locate the obstacles for students within (G', O', A') is in the set of practices and knowledge components O' which they have to acquire, as opposed to those they have previously known (O). As difficulties appear most strikingly in the setting of concrete tasks which the students experience as difficult or impossible, many studies focus on such tasks and how they relate to the global transition. Here, we shall consider the following task, and expand our analysis of it as presented in (Winsløw, 2007):

a) Show that $f(t) = t/(1+t)$ defines an increasing function on $[0, \infty)$.

b) Show that with f as above, $f(s+t) \leq f(s) + f(t)$ for all $s, t \geq 0$.

c) Show that the formula

$$d(a,b) = \frac{|a-b|}{1+|a-b|}$$

defines a metric on $\mathbb{R}$.

In fact, c) is the enunciation of a text book task (Carothers, 2000, p. 37) while a) and b) are provided as hints in the book. This is a typical task given to students as they begin to study the concept of a *metric*, defined axiomatically by three properties: on a space M, a metric is a function on $M \times M$ which satisfies, for all x, y, z in M: $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$. In the case of c), the first two properties are immediate and the last one is verified using a) and b), bearing in mind that $\delta(a, b) = |a - b|$ defines a metric on $\mathbb{R}$ (corresponding to the usual concept of distance on $\mathbb{R}$).

In actual practice, our observations of numerous exercise sessions show that students take the “hint” to rephrase the exercise as a three step procedure, as formulated above; that almost all solve part a) by computing the derivative and showing it’s positive; that few students solve b), often by round-about methods involving functions of two variables (and sometimes even a computer algebra system); and that very few students were able to make use of a) and b) to solve c), in fact in 6 out of 8 groups of 25-30 students observed, no student had succeeded to do so. Another point is that students having failed with b) did not even try to tackle c).

4.1. TDS approach. The exercise can be considered as a didactic milieu devolved by teachers to students and presenting certain obstacles, the overcoming of which are the price of the experience which the teacher aims for the students to have, of applying the definition of metrics. The first parts seem more familiar to the students and its form activate existing contracts, in the sense that surface parts of the enunciation (“increasing” and “≤”, corresponding to artefacts in the milieu) trigger certain techniques of calculation. In particular, to show that a concrete function is “increasing” one computes the derivative and “see” that it is positive. To “see” an inequality may take some rewriting; the presence of two variables in b) (again, artefacts of the milieu) is responsible for many complicated attempts to use partial differentiation and the like. Finally, no contract has been established for a “right way” of showing that something is a metric; the tripartite nature of the definition is no doubt part of the problem (one has to verify three properties instead of one). In the classroom situation, the teacher can get no further than to make the students recite (or look up) the text book definition. The teacher’s (and the text book author’s) expectation that this will be a simple experience with applying the definition thus fails because the milieu leads the students to identify the problem with contracts at the “micro”-level of the individual steps. A more global contract is visible in the fact that failure with b) kept many from even considering c), amounting to an understanding of exercises as built of from increasingly difficult parts (“if you can’t do b) then you certainly can’t do c)”). The outcome of this analysis, in the setting of TDS, is that the milieu will have to be *redesigned* to better fit the teachers’ intentions (the target knowledge, surely to be further analysed!) as well as the contractual phenomena evidenced by the students’ response to the original tasks. In particular, the properties of metrics – the key element of *O’* for the above task – may have to be (re)constructed by students first, as a response to a situation with a milieu that relates to their existing experience with (O, A).

4.2. ATD approach. Winsløw (2008) presents a two-step transition in terms of the praxeological organisations present in secondary schools and in universities: first practice blocks are completed to entire praxeologies (with theory blocks), then new practice blocks are built with tasks that take objects from "old" theory blocks. In the task above, it is mainly the second step which is in play: the function *f* and its (theoretically proved) properties are used to build a new object for a practice block related to metrics (task type: show that a given two variable function is a metric; technique: verify the axioms). Also, the "standard distance" δ (fundamental to the theory of calculus on $\mathbb{R}$) becomes one among an infinity of objects that this task type takes as an object. The institutional point of view provides a framework for explaining the apparent necessity of this two-step transition $O \rightarrow O'$. In secondary school, epistemic systems are constrained by noospheric systems pursuing aims that go much beyond the school institution itself (a range of continuing study programmes, a central examination, etc.). At universities, two types of ES coexist, those of research and those of teaching (cf. Madsen and Winsløw, to appear); the overlapping group of users consists of "professors" (research mathematicians who also teach). The complete praxeologies *O'* pursued in undergraduate mathematics programmes in research intensive universities aim to converge towards those pursued in research (*O''*). In particular, the practice of checking that a given object is a metric, as well as theories based on metric spaces, are indispensable in several branches of research mathematics. Tasks of the kind considered above are thus, at least to some extent, consequences of the *choice* that *O'* should approach *O''*.

4.3. SCT approach. Sharing some concerns with the TDS analysis presented above, a SCT analysis focuses more sharply on the beliefs and norms evidenced by the discourses found in the classroom where the exercise is discussed. While we have no space to provide even excerpt of relevant data, these might well turn out to present a cleavage in the group *G'*, between the teachers and the few students on the one hand who have formed a conception of metrics – and a technical level in algebraic manipulations – that allow for understanding and completing the task; and those students who try, quite desperately, to relate the task to norms and beliefs which they have acquired in secondary school practices. This may or may not proceed towards a progressive inclusion of the latter subgroup into a community of practice with shared norms and beliefs; but in this isolated episode, this seems to be out of reach. The fact that the majority of students did not get to consider the "real" task – because of their inability to follow the "hints" (or complete the preliminary tasks) –leads to a form of (at least) local alienation from the intended meaning-making. Such phenomena are frequently observed in studies of undergraduate mathematics within a socio-constructivist approach. Dreyfus (1999, 106) subsumes the transition which many students fail to make as moving from questions of type 'What is the result?' to questions of type 'Is it true that …?', after arguing that even within the community of mathematicians, there are no universal criteria for whether an answer to the latter type of question is complete and correct (cf. also Ernest, 1998). A SCT approach thus focuses on the process of building at least local consensus in *G'* about this matter, given that *G* has mainly been engaged in practices where it does not occur.

4.4. CST approach. Here, the first question could well to be: when addressing the three parts of the task, what are the forms of representation (including the variety of semiotic artefacts) available – actually and potentially – to students (relation of *G* to *A*)? For part a), the students can easily graph the function and thus become intuitively convinced of the claim (part of *O*). For a more formal argument, they can use the algebraic register and either differentiate f to get $(1+t)^{-2} > 0$, or use a treatment like $t/(1+t) = 1-1/(1+t)$. The latter (and simpler) *ad hoc* argument did not occur among students or even teachers, because it is not produced by an algorithm, and the standard procedure is reasonably easy. For b) there is no easy general procedure and the treatment required is equally non-standard; so only few students succeed. In fact, the complexity of required treatments *not given by a standard algorithm* also explains why the almost all students fail with c): here one has to *combine* previous results with the validity of the axioms for δ. Moreover, unlike a) and to some extent b), representations of the involved objects in other registers, such as the graphs or tables of the functions d and δ, are of no help to understand or intuitively support the sought conclusion. By contrast, in secondary level mathematics, the predominant mode of thinking involves *coordination* of several registers (e.g. graphical, symbolic, numeric) of the objects considered. While this is both a challenge and a support at the secondary level, it tends to disappear for more abstract mathematical objects at tertiary level. In the concrete case, a task on metrics on $\mathbb{R}^2$ (with ample possibilities for illustrating the different metrics) might help to enable a multimodal first encounter with the notion, on the condition that students succeed in coordinating the involved forms of representation.

5. CONCLUDING REMARKS

In this paper we compared four theories in general (see 3.5). While it would be too simplistic to maintain that the considered theoretical frameworks model only one or two of the three components of ES, they do exhibit very different foregrounds in the sense that each provides highly developed notions and principles for analysing certain components or relations between them, while leaving other in the background. One might also talk of *ontological foregrounds* in the sense that different parts of didactical reality are identified or constructed through these models. This is also illustrated by the case study (section 4). Serious integration of theoretical frameworks may eventually become possible and even useful to some extent; but I personally feel that it is more urgent to develop the rationality with which we *choose* frameworks according to *a given purpose of research*. Analysing theoretical frameworks within the GOA model may contribute to this end because research purposes and ontological foregrounds are strongly interdependent.

References.

Barbé, J., Bosch, M., Espinoza, L., Gascón, J. (2005). Didactic restrictions on teachers practice - the case of limits of functions at Spanish high schools. *Educational Studies in Mathematics 59 (1-3)*, 235-268.

Barquero, B., Bosch, M., Gascòn, J. (2006). Using research and study courses for teaching mathematical modelling at university level. In M. Bosch (ed.), *Proceedings of CERME 4*, 2050-2059. Barcelona: Universitat Ramon Llull.

Brousseau, G. (1997). *The theory of didactical situations in mathematics.* Dordrecht: Kluwer Academic Publishers.

Carothers, N. L. (2000). *Real Analysis.* Cambridge: Cambridge U. Press.

Chevallard, Y. (1985). *La transposition didactique.* Grenoble : La Pensée Sauvage.

Chevallard, Y. (2002). Organiser l'étude 1 & 3. In: J. L. Dorier et al. (eds), *Actes de la 11^{e} école de didactique des mathématiques*, pp. 3-32, 41-56. Grenoble : La Pensée Sauvage.

Cobb, P., Stephan, M., McClain, K., Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences 10*, 113-163.

Duval, R. (1995). *Semiosis et pensée humaine.* Bern: Peter Lang.

Duval, R. (2000). Basic issues for research in mathematics education. In: T. Nakahara et al. (eds.), *Proceedings of PME 24*, vol. 1, 55-69. Hiroshima: Hiroshima U.

Dreyfuss, T. (1999). Why Johnny can't prove. *Educational Studies in Mathematics* 38 (1-3), 85-109.

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics.* New York: SUNY Press.

Gueudet, G. (2008). Investigating the secondary–tertiary transition. *Educational Studies in Mathematics 67 (3)*, 237-254.

Madsen, L. and Winsløw, C. (to appear). Relations between teaching and research in physical geography and mathematics at research intensive universities. *International Journal of Science and Mathematics Education*.

Prediger, S., Bikner-Ahsbahs, Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM Mathematics Education 40*, 165-178.

Radford, L. (2008). Connecting theories in mathematics education : challenges and possibilities. *ZDM Mathematics Education 40*, 317-327.

Winsløw, C. (2004). Semiotics as an analytic tool for the didactics of mathematics. *Nordic studies in Mathematics Education 9 (2)*, 81-100.

Winsløw, C. (2007). Les problèmes de transition dans l'enseignement de l'analyse et la complémentarité des approches diverses de la didactique. *Annales de didactiques et de sciences cognitives 12*, 189-204.

Winsløw, C. (2008). Transformer la théorie en tâches: La transition du concret à l'abstrait en analyse réelle. In A. Rouchier et al. (Eds.), *Actes de la XIIIième école d'été de didactique des mathématiques* (cd-rom). Grenoble : La Pensée Sauvage.