# NAVIGATION IN GEOGRAPHICAL SPACE

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*This study is part of the ReMath project (Remath' – Representing Mathematics with Digital Media FP6, IST-4, STREP 026751 (2005 – 2008), http://remath.cti.gr. Twenty four 10th Grade students participated in a constructivist teaching experiment, the aim of which was to investigate children's constructions of mathematical meanings concerning the concept of function while navigating within 3d large scale spaces. The results showed that the utilization of the new representations provided by the dynamic digital media such as Cruislet could reform the way that mathematical concepts are presented in the curricula and possibly approach these mathematical notions through meaningful situations. The new representations provide the opportunity to introduce and study mathematical notions not as isolated entities but rather as interconnected functionalities of meaningful real – life situations.*

Functions are a central feature of mathematics curricula, both past and present. Many research studies indicate students' difficulty in understanding the concept of functions. This difficulty comes from a) the static media used to represent the concept, b) the introduction of function mainly as a mapping between sets in conventional curricula, c) the use of formalisation and function graphs as the only representations. With digital media, students can dynamically manipulate informal representations of function defined as co-variation and rate of change, which is an interesting and powerful mathematical concept. Tall(1996) points out a fundamental fault-line in "calculus" courses which attempt to build on formal definitions and theorems from the beginning. Moreover, he suggests that enactive sensations of moving objects may give a sense that "continuous" change implies the existence of a "rate of change", in the sense of relating the theoretically different formal definitions of continuity and differentiability. The enactive experiences provide an intuitive basis for elementary calculus built with numeric, symbolic and visual representations.

The 'Cruislet' environment is a state-of-the-art dynamic digital artefact that has been designed and developed within the Eu ReMath project. It is designed for mathematically driven navigations in virtual 3d geographical spaces and is comprised of two interdependent representational systems for defining a displacement in 3d space, a spherical coordinate and a geographical coordinate system. We consider that the new representations enabled by digital media such as Cruislet can place mathematical concepts in a central role for both controlling and measuring the behaviours of objects and entities in virtual 3d environments. The notion of *navigational mathematics* is used to describe the mathematical concepts that are embedded and the mathematical abilities the development of which is supported within the Cruislet microworld. In this study we focus on how students using spherical and geographical systems of reference in Cruislet construct meanings about the concept of function.

## THEORETICAL FRAMEWORK

A number of research studies suggest that students of all grades, even undergraduate students, have difficulties modelling functional relationships of situations involving the rate of change of one variable as it continuously varies in a dependent relationship with another variable (Carlson et all, 2002; Carlson, 1998, Monk & Nemirovsky, 1994). This ability is essential for interpreting models of dynamic events and foundational for understanding major concepts of calculus and differential equations. On the other hand, the VisualMath curriculum (Yerushalmy & Shternberg, 2001) is an a example of a function based curriculum that involves the moving across multiple views of symbols, graphs, and functions. VisualMath uses specially designed software environments such as simulations' software, or other modelling tools that include dynamic forms of representations of computational processes. Yerushalmy (2004) suggests that such emphasis on modeling offers students means and tools to reason about differences and variations (rate of change). Moreover, .Kaput and Roschelle (1998) using computer simulations study aspects of calculus at an earlier stage. These simulations (MBL tools), permit the study of change and the ways it relates to the qualities of the situation. In their study Nemirovsky, Kaput and Roschelle (1998) show that young children can use the rate of change as a way to explore functional understanding. In studying the process of the understanding of dynamic functional relationships, Thompson (1994) has suggested that the concept of rate is foundational.

Confrey and Smith (1994) choose the concept of rate of change as an entry to thinking about functions. They introduce introduce two general approaches to creating and conceptualizing functional relationships, a correspondence and a covariation approach. They suggest that “a covariational approach to functions makes the rate of change concept more visible and at the same time, more critical (p. 138). They explicate a notion of covariation that entails moving between successive values of one variable and coordinating this with moving between corresponding successive values of another variable.

Moreover, Carlson, Larsen and Jacobs (2001) stress the importance of covariational reasoning as an important ability for interpreting, describing and representing the behavior of dynamic function event. They consider covariational reasoning to be the cognitive ability involved in coordinating images of two varying quantities and attending to the ways in which they change in relation to each other. On the same line, Saldanha and Thompson (1998) introduced a theory of developmental images of covariation. In particular, they considered possible imagistic foundations for someone’s ability to see covariation. Carlson et all (2001) in their study exploring the role of covariational reasoning in the development of the concepts of limit and accumulation, suggest a framework including five categories of mental actions of covariational reasoning:

1. An image of two variables changing simultaneously
2. A loosely coordinated image of how the variables are changing with respect to each other
3. An image of an amount of change of one variable while considering changes in discrete amounts of the other variable
4. An image of the average rate-of-change of the function with uniform increments of change in the input variable
5. An image of the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function

The proposed covariation framework contains five distinct developmental levels of mental actions. Using this particular framework we will try to classify students' covariational reasoning while studying navigation within the context of Cruislet microworld. We consider navigation as a dynamic function event. The function's independent variable is the geographical coordinates of the position of the first aeroplane, which students are asked to navigate, while the dependent variable is the geographical coordinates of the position of the second aeroplane.

Our approach to learning promotes investigation through the design of activities that offer a research framework to investigate purposeful ways that allow children to appreciate the utility of mathematical ideas (Ainley & Pratt, 2002). In this context, our approach is to design tasks for either exclusively mathematical activities or multi-domain projects containing a mathematical element within the theme which can be considered as marginalized or obscure within the official mathematics curriculum (Kynigos & Yiannoutsou, 2002, Yiannoutsou & Kynigos, 2004).

## TASKS

In the tasks that are included in this teaching experiment, students actually engage with the study of the existence of a rate of change of the displacements of the airplanes which are defined in the geographical coordinate system. In particular the dispacements of two airplanes are relative according to a linear function. This function will be hidden and the students will have to guess it in the first phase of the activity based on repeated moves of aeroplane A and observations of the relative positions and moves of planes A and B. The second phase, the students will be able to change the function of relative motion and play games with objectives they may define for themselves such as move plane A from Athens to Thessaloniki and plane B from Athens to Rhodes and then to Thessaloniki in the same time period.

This scenario is based on the idea of half – baked games, an idea taken from microworld design (Kynigos, 2007). These are games that incorporate an interesting game idea, but they are incomplete *by design* in order to encourage students to change their rules. Students play *and* change them and thus adopt the roles of both player and designer of the game (Kafai, 2006).

Initially, students are asked to study the relation between the two aeroplanes, the rate of change of their displacements and consequently find the linear function (decode the rule of the game). In order to decode "the rule of the game", they should give various values to coordinates (Lat, Long, Height) that define the position of the first plane. They will be encouraged to communicate their observations about the position of the second plane to each other and form conjectures about the relationship between the positions of the two aeroplanes.

In the second phase students are encouraged to build their own rules of the game by changing the function of the relative displacements of the two aeroplanes.

## METHODOLOGY

The research methodology is a constructivist teaching experiment along the same lines as described by Cobb, Yackel and Wood (1992).The researcher acts as a teacher interacting with the children aiming to investigate their thinking. The researcher, reflecting on these interactions, tries to interpret children's actions and finally forms models-assumptions concerning their conceptions. These assumptions are evaluated and consequently either verified or revised.

Twenty four (24) students of the 1st grade of upper high school, (aged 15-16 years old) participated in this experiment. Students worked in pairs in the PC lab. Each pair of students worked on the tasks using Cruislet software.

The data consists of audio and screen recordings as well as students' activity sheets and notes. The data was analyzed verbatim in relation to students' interaction with the environment. We have focused particularly on the process by which implicit mathematical knowledge is constructed during shared student activity. As a result, in our analysis we use students' verbal transcriptions as well as their interaction with the provided representations displayed on the computer screen.

## ANALYSIS

While students were interacting with the Cruislet environment according to the tasks, several meanings emerged regarding the concept of function. We categorise these meanings in clusters that rely upon the concept of function. In particular, there are two major categories:

### Domain of numbers

Students navigating an aeroplane in the 3d map of Greece realized that the domain of the geographical coordinates is actually a closed group. The 3d map of Greece is a geographical coordinate system with specific borders. The investigation of the range of the geographical borders as the domain of the function became the subject of study and exploration through the use of the Cruislet functionalities. In particular, students exploited the two different systems of reference and, experimenting with the values of the geographical coordinates, they define the range of the latitude – longitude values. This specific range of values has been considered as the domain of the functions according to which the displacements of the aeroplanes are relative. Although students didn't refer to the values as the domain of the function, we interpret their involvement in finding them, as a mathematical activity regarding the domain of the function.

Students experimented by giving several values to the geographical coordinates of the airplane's position defining at the same time the range of the coordinates' values. In the following episode students are trying to find out the reason for not placing the airplane in a given position.

S1: Why?? It doesn't accept any value. (they gave values in procedure fly1 and the airplane couldn't go).

R: Do you remember what values the lat long coordinates have?

Isn't lat equals 58 isn't correct? (she also speaks to the next team)

S1:It doesn't accept 32 20 100 either.

S2: Greece hasn't got value 20 (student from another team speak ironically to him)

S1: Why? Was the 58 you used correct?

An interesting issue related to the domain of the function, is that the provided representations, i.e. the result of the aeroplane's displacement displayed on the screen, helped students realize that the domain of numbers of the two aeroplanes displaced in relative positions, are strongly dependent. For instance when the first moved to a given position, the second one couldn't go anywhere, but the domain of values was restricted by the first position. In the following episode students realized that the 2nd aeroplane didn't follow them when they flew at a low height. The episode is interesting as it indicates the way students realize the domain of geographical coordinate values that the first aeroplane can take in relation to the other one.

S1: There are some times that it (meaning the other aeroplane) can't follow us.

R: Where? When?

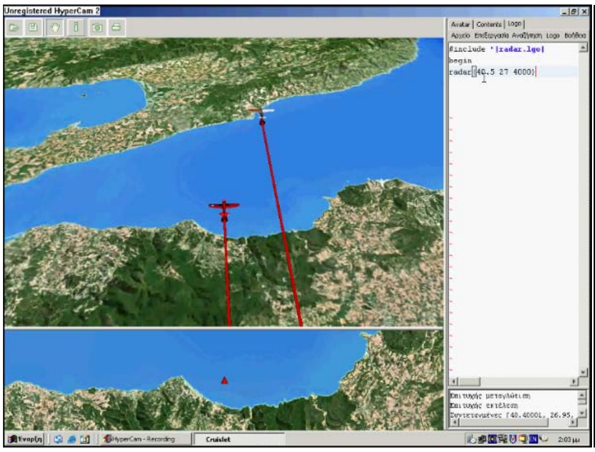
S1: When I'm getting into the sea.

We could say that the characteristics of Cruislet software, such as the visualization of the results of the objects' displacements on the map, acted as a mediator in students' engagement with the domain of function. We have to mention that although the modalities of use of Cruislet software and the communication within the groups didn't reveal that students realized or mentioned anything regarding the concept of function, they did focus on finding ways to move the aeroplanes. In other words, students didn't conceive the values of the coordinates as the domain of the function, although they used it in this way. The interpretation of students' actions relies upon our educational goals, which conceive this as a mathematical activity that was related to the notion of function and particularly, its domain.

**Function as covariation**

During the implementation of the tasks, students engaged with the notion of function, through their experimentation with the dependent relationship between two aeroplanes' positions, which was defined by a black – box Logo procedure. Trying to find out the hidden function, students' actions and meanings created, suggested they were able to coordinate changes in the direction and the amount of change of the dependent variable in tandem with an imagined change of the independent variable. Our results indicate that students developed covariational reasoning abilities, resulting in viewing the function as covariation.

Initially most of the students expressed the covariation of the aeroplanes' positions using verbal descriptions, such as behind, front, left, etc. as they were visualizing the result of the airplanes' displacements. In the following episode students express the dependent relationship while looking at the result displayed on the screen.



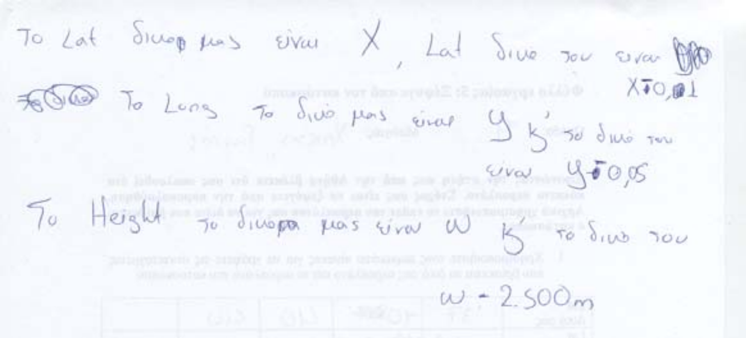
S1: So, he always wants to be close to us on our left.

R: Yes.

S1: And he is beneath, further down to us. Beneath.

S2: And behind.

Students experimented by giving several values to geographical coordinates in Logo and formed conjectures about the correlation between the aeroplanes' positions. Through their interaction with the available representations, they successfully found the dependent relation of the function in each coordinate, resulting in their coming into contact with the concept of function as a local dependency. In fact, one of the teams conceived the relationship among each coordinate as a function, as is obvious in their notes on the activity sheet.


Το Lat δικό μας είναι X, Lat δικό του είναι X-0,1
Το Long το δικό μας είναι y κ' το δικό του είναι y-0,05
Το Height το δικό μας είναι ω κ' το δικό του ω-2.500m

Translation

| |
|---|
| Our Lat is x, his Lat is x – 0.1 |
| Our Long is y and his is y – 0.05 |
| Our Height is ω and his is ω – 2500m. |

It is interesting to mention that students separated latitude and longitude coordinates on the one hand and that of height on the other as they were trying to decode the hidden functional relationship between the airplanes' height coordinates. In particular, they didn't encounter difficulties in decoding latitude and longitude relationship in contrast to their attempts to find the height dependency. Although all three functions regarding coordinates were linear, students conceived the functional relationship between height mainly as proportional, in contrast to latitude and longitude that were comprehended as linear, from the beginning. In the following episode, students endeavor to apply the rate of change of the function to decode the height relationship. As they thought the height coordinates had a proportional relationship, they suggested carrying out a division to find it.

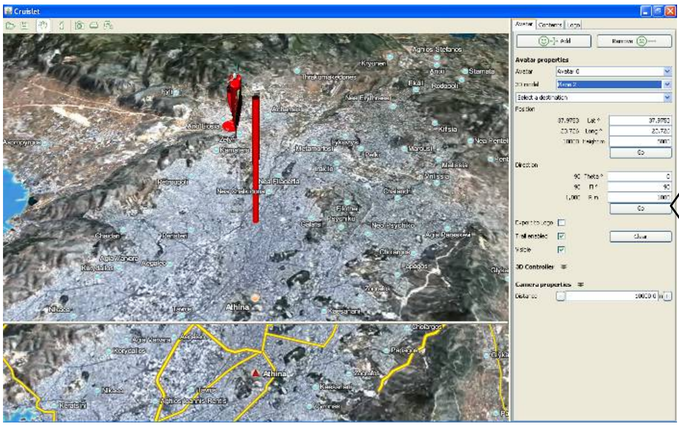
S2: When we go up 1000, he goes up 1000.

R: Do you mean that if we go from 7000 to 8000 he goes from… let's say 2500 to 3500.

S2: He is at… 3000. No. Give me a moment. At 8000 he was at 5500. At 7000 he was at 4500. At 5000 he is as 2500. And then….

S1: We could do the division to see the rate.

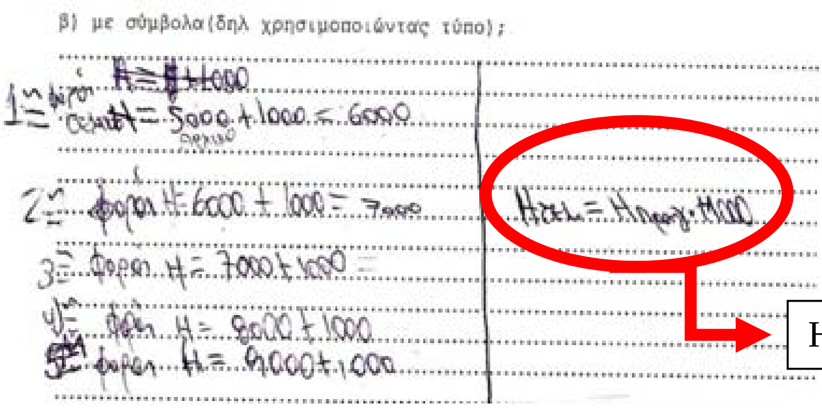
An interesting example was the cases of the variation of the height of the aeroplane every time they pushed the button 'go' in spherical coordinates, when they wanted to make a vertical displacement. In particular, by defining the vector of a vertical upward displacement, students observed that height was the only element that changed in the position of the displacement. Through a number of identical displacements students identified and expressed verbally, symbolically and graphically the interdependency between direction functionality and the height of the aeroplane. Students' reasoning: "the more times we push the button GO the higher the aeroplane goes", suggests that students developed a covariational reasoning ability similar to the second level proposed by Carlson et al (2001) of how the variables change with respect to each other. Moreover, the retrospective symbolic type developed by students (h2= h1+1000) indicates that they realized that the rate of change of the height is constant. In the following figures we can see the result displayed on the screen (figure 1) as well as students' writings on the activity sheet (figure 2).



Students' actions:

1. Define the spherical coordinates (theta = 0 fi = 90 R = 1000).
2. Push the "Go" button in "Avatar properties" tab resulting in the vertical displacement of the aeroplane.
3. Watch the displacement of the aeroplane on the GUI.
4. Focus on the changes of the height coordinate.

Figure 1




$H_{final} = H_{before} + 1000$

Figure 2

The provided representations of Cruislet software became a vehicle to engage students with concepts related to the concept of function and their expression in a mathematical way. The result of airplanes' displacements on the screen, gave them the chance to realize the dependent relation in 'visual terms' and then express it in mathematical terms. We believe that the results are mainly based on the way that these characteristics were used in the task activity. In particular, the activity was based on the idea of the 'Guess my function' game and the dependent relationship, (built in Logo programming language), was hidden at first. Due to this choice, students focused primarily on the observation of the relative displacements and not on the Logo code underneath it. At the same time perceiving the activity as a game encourages the engagement of students with the activity.

## CONCLUSIONS

The study indicated that students exploiting Cruislet functionalities can construct meanings concerning the concept of functions. The provided linked representations (spherical and geographical coordinates), as well as the functionalities of navigating in real 3d large scale spaces actually enable students to explore and build mathematical meanings of the concept of function within a meaningful context. They explore the domain of numbers of a function within a real world situation distanced from the "traditional" formal definitions. On the other hand, they built the concept of function as covariation exploring the variation of the spherical and geographical coordinates. The provided context gave students the opportunity to cope with and explore mathematical concepts at different levels. They navigate within 3d large scale spaces controlling the displacement of an avatar and develop their visualization abilities building mathematical meanings of the concept of function while at the same time they explore the mathematical concepts of spherical and geographical coordinates.

The functionalities of the new digital media such as Cruislet provide a challenging learning context where the different mathematical concepts and mathematical abilities are embedded and interconnected. The role of the teacher becomes crucial in designing mathematical tasks where students' enactive explorations will reveal these mathematical notions and put them under negotiation. In the case of Cruislet, navigational mathematics becomes the core of the mathematical concepts that involves the geographical and spherical coordinate system interconnected with the concept of function and the visualization ability.

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