A LEARNING ENVIRONMENT TO SUPPORT MATHEMATICAL GENERALISATION IN THE CLASSROOM

Geraniou, E., Mavrikis, M., Hoyles, C. and Noss, R.

London Knowledge Lab, Institute of Education, UK

This paper discusses classroom dynamics and pedagogical strategies that support teaching mathematical generalisation through activities embedding a specially-designed microworld. A prototype of our microworld was used during several one-to-one and classroom studies. The preliminary analysis of the data have allowed us to see the implications of designing and evaluating this specific technological tool in the classroom as well as the teachers' and the students' requirements. These studies feed into the design of the intelligent support that we envisage the system will be able to offer to all students and the teacher. In particular, they helped us identify which aspects of teachers' interventions could be delegated to our system and what types of information would be useful for supporting teachers.

KEYWORDS: Mathematical Generalisation, Microworlds, Classroom Practices, Teachers, Intelligent Support

INTRODUCTION

It seems that there is a growing diversity of computer-assisted material and tools for mathematics classrooms. Even though this proliferation of digital tools and new technologies has broadened the instructional material available for teachers, they are still rather insignificant to classroom practice and their use is far from regular (Artigue, 2002, Mullis et al., 2004, Ruthven, 2008). This suggests a challenge for mathematics educators to develop complete, consistent and coherent systems that not only assist students, but also support teachers' practice in the classroom.

The aim of the MiGen[1] project is to design and implement a system with teachers that meets their as well as students' requirements. We are developing an intelligent exploratory learning environment for supporting students in making mathematical generalisations. In more detail, our focus has been on the difficulties, first students face in their efforts to generalise and second teachers face in their efforts to support students appropriately during lessons with 20-30 students. For our initial investigations, we restricted the domain of mathematical generalisation to the generation and analysis of patterns. Activities with patterns often appear in the UK mathematics curriculum and have been identified as motivating for students (see Moss & Beatty, 2006). They also comprise a good domain for generalisation, since they allow students to come up with different constructions for the same pattern, find the corresponding rules and realise their equivalence.

Our aim is to develop a system that provides the means to understand the idea of generalisation, but also the vocabulary to express it, while supporting rather than supplementing the teacher. The system is intended to provide feedback to the teacher about their students' progress and, where the system's 'intelligence' is unable to help students, to prioritise the students in critical need of the teacher's assistance.

The core of our system[2] is a microworld, called the eXpresser (described briefly in the next section), in which students can construct and analyse general patterns using a carefully designed interface. In order to build the microworld, our team[3] started with a first prototype (Pearce et. al, 2008). Using an iterative design process, and in order to investigate the effectiveness of our approach, we carried out a number of studies with individual students or pairs of students, each time using the feedback we obtained to build the next prototype. This process resulted in the evolution of the prototype and its subsequent evaluation in classroom.

This paper, after a brief discussion of our methodology, presents the preliminary data analysis of the classroom studies that not only support the next version of the microworld, but also feed into the design of the intelligent support that we envisage the system will be able to provide. Our focus here is on the teachers' pedagogical strategies and the students' needs for support and assistance during their interactions with the microworld. This analysis is followed by a discussion of the teachers' interventions that could be delegated to the 'intelligent' system and what types of information would be useful for supporting teachers and therefore necessary for the development of the intelligent support components of our and other similar systems.

A microworld for patterns – the eXpresser

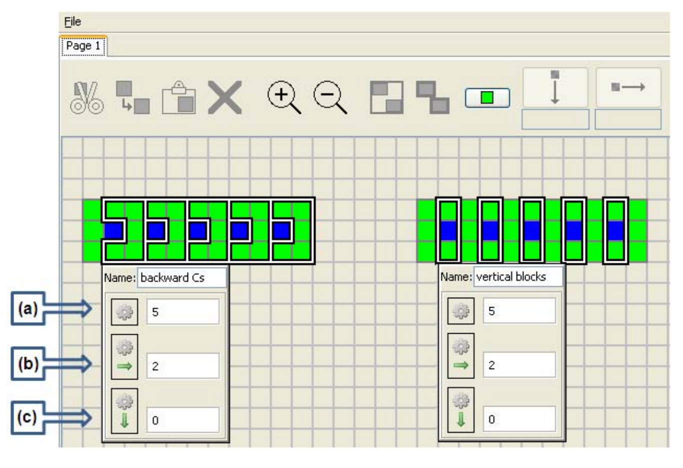

Figure 1. The interface of eXpresser with two different constructions of the same pattern. The left one is made out of a vertical block of 3 squares and 5 'backward C-s' and the right one of alternating vertical blocks.

First, we present briefly the main features of the eXpresser. We emphasise that at the stage of the study, attention was focused largely on the features key to our research goals. So, the following description of the system is by no means complete. In addition, its design has evolved significantly through studies such as the ones described in this paper. The interested reader is referred to Noss et al. (2008), where the system's rationale and design principles are described in detail.

In eXpresser, students can construct patterns based on a 'unit of repetition' that consists of square tiles. These patterns can be combined to form complex patterns, i.e. a group of patterns. A pattern's property box (depicted in Figure 1) shows three numeric attributes that characterise the pattern[4]. The first specifies the *element count* (number of repetitions) of this pattern (a). The icon with the right arrow (b) specifies *how far* *to the right* each shape should be from its predecessor and, similarly, the icon with the down arrow (c) specifies *how far down* a shape should be.

A requirement of our constructivist approach was to allow students to construct patterns in a variety of ways (Figure 1). Additionally, an important design feature is the ability to 'build with n' (see Noss et al., 2008), i.e. to use independent variables of the task to create relationships between patterns.

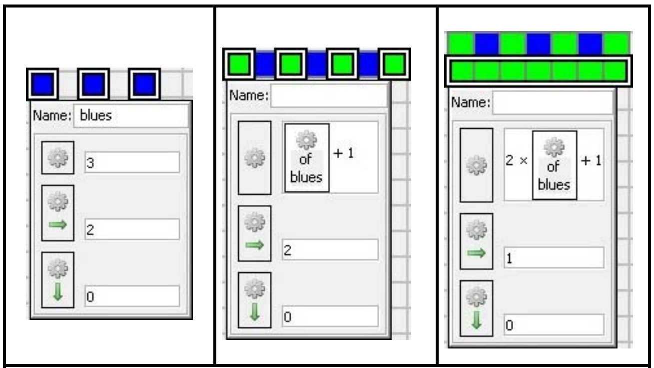

Figure 2. Another way to construct the pattern in Figure 1. To relate the middle row with the first pattern (named "blues"), the number of repetitions should be one more than the number of repetitions of "blues". For the bottom row it should be twice more plus one. These relationships are specified iconically.

This feature not only provides students additional ways to construct patterns but we hypothesised that it enables students to realise what are the independent variables and use them to express relationships. To overcome difficulties that students face with symbolic variables the microworld employs what we call 'icon-variables', which are pictorial representations of an attribute of their construction. We have illustrated in previous work (Geraniou et al., 2008), that these 'icon-variables' provide a way to identify a general concept that is easier for young learners to comprehend. An example of expressing such relationships is depicted in Figure 2.

METHODOLOGY

Our own previous work and studies by Underwood et al. (1996) and Pelgrum (2001), for example, concerning the adoption of educational software in classrooms emphasise the importance of teachers' involvement in the whole design process of computer-based environments. Therefore, several meetings with the teacher were held before each classroom session so that they were familiarised with the prototype, agreed and made input to the lesson plans and in order to clearly state the teacher's, the students' as well as the researchers' objectives.

The overall methodological approach is that of 'design experiment', as described by Cobb et al. (2003). One of our goals during these sessions was to inform our system's design and evaluate the effectiveness of our pedagogical and technical approach. We aimed at investigating the classroom dynamics by looking at individual students' interactions with the microworld, the collaboration among pairs or groups of students as well as the teachers and researchers' intervention strategies.

We investigated the use of eXpresser in several one-to-one and classroom sessions with year 7 students (aged 11-12 years old). Particularly for the classroom sessions, two researchers played the role of teaching assistants and another was observing and keeping detailed notes regarding the researchers' and the teacher's interventions. The sessions were recorded on video and later analysed and annotated with the help of the written observations. Based on these, we were able to get information regarding the time and duration of the interventions, the type of feedback given, the students' reactions and immediate progress after the interventions. Therefore, our goals in the study reported in this paper were to identify not only the students' ability to collaborate successfully and articulate the rules underpinning their generalisation of the patterns but particularly when and how the teacher or the researchers intervened.

However, to maintain the essence of exploratory learning, research suggests a teacher's role should be that of a 'technical assistant', a 'collaborator' (Heid et al., 1990), a 'competent guide' (Leron, 1985) or a 'facilitator' (Hoyles & Sutherland, 1989). Our aim was to achieve the right balance between students' autonomy and responsibility over their mathematical work and teachers' and researchers' efforts to scaffold and support their interactions. The teacher and the researchers set out to adopt this role by following a specific intervention philosophy that adhered to our framework of interventions (Mavrikis et al., 2008), which was based on our previous work with Logo and dynamic geometry environments. This framework was extended after the analysis of the data and is presented in the 'Classroom Dynamics' section. Our aim was to avoid imposing our (or the teacher's) views or ways of thinking, but instead allowing students to express their viewpoints and assist them by demonstrating the tools they could use: for example, by directing their attention, organising their working space and monitoring their work.

CLASSROOM SCENARIO

We illustrate here a classroom scenario carried out with a year 7 class with 18 high-attaining students. Students were introduced to the microworld through a familiarisation process, during which the teacher introduced all the key features to construct a simple pattern and students followed his actions on their laptops.

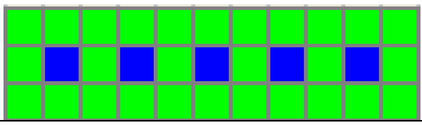

Figure 3. The activity: Find a rule for calculating the number of green (light) tiles for any chosen number of blue (dark) ones.

Students were then presented with the task in Figure 3. The pattern was shown dynamically on the whiteboard; its size changed randomly showing a different instance of the pattern each time. This made it impossible for students to count the number of tiles while allowing them to 'see' variant and invariant parts of the pattern. We hypothesised that a dynamically presented task would discourage 'pattern-spotting', which focuses on the numeric aspect of specific instances of the pattern, and counting, which encourages constructing specific cases of the pattern. It also provided a rationale for the need of a general rule that provides the number of tiles for *any* instance of the pattern.

Students were given the freedom to construct the pattern in their own way, using the system's features they had been shown earlier. They were asked to write on a hand-out how they constructed the given pattern and then discuss in pairs their constructions and the methods they followed. They also worked collaboratively to find a rule that gives the number of green tiles for any chosen number of blue ones. Students' next challenge was to find different ways to replicate the pattern and describe them on the hand-out explicitly, so as their partner could understand it. After discussing with their partner, if they had come up with the same constructions, they were expected to try to see whether there were any other ways and find all the rules that represented their constructions and write them down. Finally, the teacher initiated a discussion, where students were asked to present their rules to the rest of the class. Rich arguments were developed and students challenged each other to justify the generality of their construction and the rules they have developed.

During this classroom study many interesting issues regarding the classroom dynamics were identified that informed our further design of the microworld and the overall system and the next phase of the research.

CLASSROOM-DYNAMICS

As expected, to ensure the success and effectiveness of students' interactions with the eXpresser, there was a need for significant support from the teacher and the researchers. As discussed already, we had agreed a specific intervention philosophy with the teacher. The analysis of the data (video recordings and written observations) revealed further strategies and extended our previous framework of interventions (Mavrikis et al., 2008). The revised framework is presented in Table 1.

- Reminding students of the microworld's affordances
- Supporting processes of mathematical exploration
  - *Supporting students to work towards explicit goals*
  - *Helping students to organise their working environment*
  - *Directing students' attention*
  - *Provoking cognitive conflict*
  - *Providing additional challenges*
- Supporting collaboration
  - *Students as 'teaching assistants'*
  - *Group allocations*
  - *Encourage productive discussion (group or classroom)*
- Ensuring task-engagement and promoting motivation

Table 1. Types of interventions observed during our studies

Below we pull out some illustrative episodes under each category.

Reminding students of the microworld's affordances

As facilitators the teacher and the researchers (referred to as 'facilitators' for the rest of the paper) managed to support students' interactions and explorations by reminding them of various features of the system that assisted students' immediate goals. This intervention acted sometimes as a prompt and other times as an offer of assistance. If the facilitator sensed a student was working towards a direction where they could be assisted by a specific tool, they would point it out to their students. This teaching strategy might have proved rather common as for some students the one lesson spent on familiarisation with the system seemed not enough.

Supporting processes of mathematical exploration

We often needed to support the students' problem-solving strategies. For example, we noted that students tended to forget their overall goal. Students seemed to get lost in details and got carried away with various constructions ('drawings'), which, even though offering students more experience of the system's features and affordances, it sometimes led them in the wrong direction. One of the downsides of any microworld is that students' actions can become disconnected from the mathematical aspects under exploration. Even though, the system's affordances were carefully designed to support students' thinking processes, they were not always naturally adopted by them. Therefore, when needed, we provided a reminder of their goals or helped them re-establish them by asking questions like "What are you trying to do?" or "What will you do next?" (*supporting students' work towards explicit goals*).

Another aspect of problem-solving skills (particularly when working in microworlds) that some students seemed to lack was being able to come up with *an organised working environment*. We occasionally advised students to delete shapes that were irrelevant to the solution or change the location of a shape so that they could concentrate on ones that could prove useful. It was evident that students who worked effectively and reached their goals were the ones that organised their working space and therefore supported their perception of the task in hand.

Directing students' attention was a necessary pedagogic strategy. We prompted students to notice invariants or other details which are important for their investigations without giving away the answer. For example, we asked questions such as "Did you notice what happened when you increased the length of this pattern?" or "when you changed this property of your pattern?". These pointed out certain facts that students might have missed out or ignored, but also exposed possible misconceptions and misinterpretations. If students were focusing on or manipulating unnecessary elements of their construction, the facilitators provided hints towards more constructive aspects. If students' responses revealed any misconceptions, then such a prompt acted as an intervention for *provoking cognitive conflict*. There were cases where the cognitive conflict was not obvious to the students directly and further explanations were required from the facilitators. These normally involved giving counter-examples to provoke students' understanding and challenge their thinking processes. Besides this intervention we used another strategy, referred to as "messing-up", used in our previous work in dynamic geometry (Healy et al., 1994). This strategy challenged students to construct a pattern that is impervious to changes of values to the various parameters of the tasks. Students tended to construct patterns with specific values and had their constructions 'messed-up' when the facilitators suggested: "What happens when you change this to say 7 (a different value to the student's chosen one)?". This strategy gave a rationale for students to make their constructions general by encouraging them to think beyond the specific case. In other cases where students seemed to have reached a satisfactory general construction, the facilitators intervened by *providing additional challenges*. For example, "Could you find another way of constructing the pattern?".

Supporting collaboration

Students who achieved a seemingly general construction and found a rule (general or not, representing their construction or not), often failed to find different ways of constructing the pattern. Our approach in these circumstances was to introduce them to the collaborative aspect of the activity, in which they had to discuss, justify and defend the generality of their constructions and their rules to their partners. We envisaged that learners' general ways of thinking would be enhanced by the sharing of their different perspectives. Accompanied by the facilitators' or fellow students' assistance, students could appreciate the equivalence of their approaches and possibly adopt a more flexible way of thinking. In this study, the rationale behind collaboration was to give students an incentive to enrich their perception and understanding of the given pattern, to find more ways of constructing it and begin to appreciate their equivalence mathematically. The allocation of students to groups aimed at ensuring the best possible collaboration (*group allocations*). Ensuring though that discussions carried out within the groups were fruitful was not an easy task. The first step towards this goal was grouping the students in a way that promoted participation from all members of the group while discouraging students from dominating a discussion (*encourage productive discussion*).

On some occasions, the facilitators, particularly the teacher who has better insights into his students' competence, encouraged students to take the role of a *'teaching assistant'* and help others who were less successful in their constructions. This intervention boosted students' confidence, but also gave them an opportunity to reflect upon their actions and an incentive to explain their perspective.

Ensuring engagement and promoting motivation

Finally, although the activities and the system affordances were designed to assure engagement as well as promote students' motivation, there were various occasions (e.g. being stuck or 'playing' by drawing random shapes) when the facilitators' intervention was required. Our vision was to give the right rationale for students to solve the task and praise their efforts. These studies supported our view that avoiding tedious activities that were pointless in the students' eyes, not only reduces the risk of off-task behaviour, but also sustains a productive atmosphere for students.

TOWARDS AN INTELLIGENT SYSTEM IN THE CLASSROOM

The interventions that were discussed above require an intensive one-to-one interaction with the students who require help. However, it is unrealistic to expect teachers in classrooms to be able to adhere to the demanding role of facilitators, keeping track of all students' actions while allowing them to explore and have the freedom to choose their immediate goals. As mentioned above, there are multiple ways of constructing a pattern and therefore multiple ways of expressing general solutions for such activities. It is at this point that the value of a system that can provide information to the teacher becomes apparent.

One of the most practical issues regarding students' interactions in such environments is that despite the familiarisation process, there is a need to remind students of certain features or even prompt them to use those which could prove useful for their chosen strategy. Therefore, it should be possible to identify (based on students' actions) which tasks of the familiarisation activity they should repeat. An intelligent system could highlight tools relevant to their current actions or offer a quick demonstration directly taken from their familiarisation activity. Furthermore, it could repeat their previous successful interactions relevant to the current activity.

In terms of the teachers' responsibility to attend to and help all the students in a classroom our studies highlighted the difficulty to prioritise which student to help. It is inevitable, therefore, sometimes to offer support to students who do not need it as much as others or even leave some students unattended due to the time constraints of a lesson. Moreover, it is possible for students to misunderstand certain concepts and leave a lesson with a false sense of achievement. Of course, it is difficult for an intelligent system to detect this accurately. However, it is possible to draw the teacher's attention to students potentially in need. By providing therefore information regarding students' progress at various times during a lesson as well as alerting them of likely misconceptions, it becomes possible for the teacher to spend their time and effort efficiently.

Besides these teachers' difficulties, there are situations when, despite having carefully-planned lessons, teachers are required to take immediate and effective decisions during lessons to accommodate their students' needs. For example, noticing when students are having difficulty with certain tasks or providing extension work are interventions which could be delegated to our system, allowing more time for teachers to provide essential help. Moreover, the collaborative component of an activity could be supported by the system by recommending effective groupings of students and allowing them to co-construct patterns whilst reducing dominance and promoting successful collaboration. The system could inform the teacher about the dynamics of different groups and alert them of possible concerns regarding the groups' progress as well as suggest more productive groupings (e.g. group students with different constructions but equivalent general expressions).

In addition, although we acknowledge the strong dependency between motivation, engagement and the design of the activities, it was evident that some students were at points disengaged. Even if off-task behaviour can sometimes lead to fruitful outcomes and intrigue students' thinking processes towards a direction, there is a need in automatically detecting such behaviour and informing the teacher. It then becomes the teacher's responsibility to decide how and whether to intervene.

The aforementioned suggestions for intelligent support could ease the use of an exploratory environment like the eXpresser in the classroom. It is often the case that such systems end up being used as a tool just to demonstrate certain mathematical concepts because of similar difficulties faced in classroom as those we reported here. Moreover, although quite a few 'intelligent' tutoring systems have been designed to provide support and personalised feedback to students and are starting to be integrated in classroom (Forbus et al., 2001), they usually scaffold the students with predetermined solution methods and by definition restrict students' reaching their own generalisations. Our team's challenge is to build a system that provides students the freedom to explore, make mistakes, get immediate feedback on their actions while assisting teachers in their difficult role in the classroom and therefore enable the successful teaching and learning of the idea of mathematical generalisation.

NOTES

1. See http://www.migen.org/ for details. Funded by the TLRP, e-Learning Phase-II; Award no: RES-139-25-0381.

2. Our system comprises of two additional components, the eGeneraliser, which aims to provide students with personalised feedback and support during their interactions with the microworld, and the eCollaborator, which aims to foster an online learning community that supports teachers in offering their students constructions and analyses to view, compare, critique and build on.

3. We would like to acknowledge the rest of our research team and particularly Sergio Gutierrez, Ken Kahn and Darren Pearce who are working on the development of the MiGen system.

4. Each attribute has an associated icon tentatively depicted as cogs "to indicate the inner machinery of a pattern". As the design of eXpresser is evolving our team is evaluating the appropriateness of these icons.

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