CERME 6 – WORKING GROUP 5
GEOMETRICAL THINKING

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INTRODUCTION
GEOMETRICAL THINKING

Kuzniak Alain
Laboratoire André Revuz, University Paris-Diderot France
Elia Iliada
University of Cyprus
Hattermann Matthias
University of Giessen, Germany,
Filip Roubicek
Institute of Mathematics, Academy of Sciences of the Czech Republic

The Working Group 5 on Geometrical Thinking had around 30 participants from 14 countries all over Europe and from America too (Mexico, USA and Canada). During its sessions, the participants discussed 16 papers prepared for the Working Group and selected among 23 initial proposals and 15 have been retained for publication. The participants, and it's a strength of the group, worked within the continuity of the former sessions of Cerme. Some points can be considered as a common background known by ancient participants to the Working Group and the discussions among people were facilitated by this common culture. The readers are invited to have a look on the former general reports made at Bellaria (Dorier et al., 2003) and Larnaca (Kuzniak and al, 2007) when they want to know more about the common background.
This report insists on the questions of theoretical supports in Geometry, which can be seen as local theory in comparison of more general theoretical frameworks used in Mathematics Education. It would be interesting to explore the relationships between both local and global viewpoints. This part results from a collective work of a small group managed by Iliada Elia.
Then, all the accepted papers are briefly introduced for giving an idea of problems the group was concerned by.

Theoretical and methodological aspects of research in geometry

During the working group, we distinguished two approaches of using theory in research: First, theory can serve as a starting point for initiating a research study. For instance, the need to empirically validate or extend specific theories may motivate an investigation. Second, theory can act as a lens to look into the data. For example, different phenomena and behaviours observed in mathematics classes may evoke ideas to the teacher or the researcher for starting research. To start from phenomena or data is a valid first approach to research. In this case, theory may enable the teacher or the researcher to better understand and interpret the collected data.
Certainly, if one has a dual approach to research (data or theory) s/he can start with theory or data. This has methodological implications, that is, the methodology has to be appropriate to a chosen theory or to the collected data. The collection of data is

very important, though, for both types of research. But to have substantial and long-standing effects to the research community's endeavour, the data, their use and interpretation should have a theoretical contribution (e.g. add or suggest modifications to an existing theory or develop new theory).

The most important theories in geometry education that were identified and discussed are the following: Van Hiele's levels, Geometrical Working Space and Geometrical paradigms and Duval's semiotic approach. Each line of theory approaches geometry learning from a different perspective and thus is helpful for different purposes. Van Hiele's theory is mainly helpful for evaluating students' reactions, productions and solutions to problems (phenomenological approach). Houdement and Kuzniak's (2003) theory about Geometrical Working Space and Geometrical Paradigms (e.g. Geometry I: Natural Geometry, Geometry II: Natural Axiomatic Geometry and Geometry III: Formal Axiomatic Geometry) is mainly helpful for classifying approaches, e.g. the types of argumentation used and to understand students' difficulties and errors (epistemological approach). Duval's (2005) theory is mainly helpful for examining the registers (e.g. geometrical figures, verbal representations-language) used in the field of geometry and their treatment in geometry tasks (semiotic approach).

Furthermore, there are psychological approaches to geometry that are often linked to spatial abilities, e.g. Gestalt and Piaget's theories, but are not very well taken into account in the mathematics education research community. Connecting these approaches with geometry theories and/or using them as a tool to look into the data in future studies could be a first step towards addressing this gap.

Future research on geometry theories and their articulation could use Geometrical Paradigms in a more operationalized manner to analyze existing curricula, to analyze students' behaviour and in investigating modelling and problem solving. Van Hiele's levels could be extended by proposing and empirically validating new (sub-)levels within their scale.

Educational goals and curriculum in geometry

The discussion on this general and fundamental topic was introduced by two papers. Using an epistemological approach, Boris Girnat criticized some present approaches in the learning of Geometry (especially in Germany) which leave aside the classical ontological aspect of Geometry. . He claims that there are two different types of applications in geometry and that they both are necessary and not exchangeable by each other: The first one contains simple applications which are paradigmatic examples to learn basic geometrical concepts; the second one includes more complex ones and refers to transcendental aspects.

Then Laurent Vivier and Alain Kuzniak described a French viewpoint on the Greek Geometrical Work at Secondary level. Beyond some similarities between France and Greece, it appears that the Euclidean tradition stays stronger in Greece but only for cultural reasons. Due to the lack of evaluation at the entrance on the university, the teaching of geometry is not viewed as important by the students and we can notice again the effects of evaluation on the real curriculum. In their study, the authors used a theoretical frame based on paradigms and geometrical working spaces and Greek people present in the group reacted and agreed with the conclusions. The presentation made at Cerme was thought as an important part of the research project.

Understanding and use of geometrical figures and diagrams

The study presented by Eleni Deliyianni investigated the role of various aspects of apprehension, i.e., perceptual, operative and discursive apprehension, in geometrical figure understanding. Based on a statistical exploration of data collected from 1086 primary and secondary school students, the existence of six main factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concepts. However, findings revealed differences between primary and secondary school students' performance and in the way they behaved during the solution of the tasks.

In her presentation Claudia Acuna used the old but always pertinent viewpoint on the treatment of geometric diagrams as Gestalt configurations. In geometry, the figural aspects of diagrams as symbols are used to solve problems. When figural information are treated, Gestalt configurations emerge: auxiliary figural configurations, real or virtual, that give meaning and substance to an idea that facilitates the proof or solution to the problem. In the paper, some arguments are given to acknowledge the existence of these resources.

Understanding and use of concepts and "proof" in geometry.

The work presented by Paola Vighi is concerned by the comparison of surfaces which need some mereological transformations in the sense of Duval. The same problems were given to two groups of pupils 10-11 years old having followed different ways of learning geometry: one traditional and the second more "experimental". She concludes with some observations about teaching geometry and suggestions for its improvement.

Caroline Bulf studied some symmetry's effects on conceptualization of new mathematical concept at two different levels at French secondary school, with students who are 12-13 years old and 14-15 y.o. From the study, the concept of symmetry makes students confused with the transformations of the plan introduced at the beginning of secondary school. Students seem to be more familiar with metrical properties relative to the symmetry and develop mathematical reasoning at the end of secondary school.

Mattheou Kallia investigated the basic geometrical knowledge of students of the Pedagogical Department of Education. She investigated mainly how they define similarity of shapes and how the intuitive knowledge affects their perception of similar shapes. The results showed that a large percentage of students are not in a position to correctly define the similarity of shapes and that initial intuition affects their responses and their mathematical achievement.

Two other papers were focused on the question of geometrical reasoning. Georgia Panoura and Athanasios Gagatsis underlined that the geometrical reasoning of primary and secondary school students can be compared mainly on the way students confronted and solved specific geometrical tasks: the strategies they used and the common errors appearing in their solutions. This comparison shed light to students' difficulties and phenomena related to the transition from Natural Geometry (the objects of this paradigm of geometry are material objects) to Natural Axiomatic Geometry (definitions and axioms are necessary to create the objects in this paradigm of geometry). They stressed the inconsistency of the didactical contract implied in primary and secondary school education and they conclude on the need for helping students progressively move from the geometry of observation to the geometry of deduction.

Based on a different framework, Taro Fujita seems to study the same problem in the case of geometry in Japan. This paper reports findings that indicate that as many as 80% of lower secondary age students can continue to consider that experimental verifications are enough to demonstrate that geometrical statements are true - even while, at the same time, understanding that proof is required to demonstrate that geometrical statements are true. Further data show that attending more closely to the matter of the 'Generality of proof' can disturb students' beliefs about experimental verification and make deductive proof meaningful for them. It could be interesting to interpret these results with the same tools as Panoura and Gagatsis: didactical contract and geometrical paradigms. It seems that the conclusions are very close but in different context.

Communication and assessment in geometry

In the two following papers, original tools were used to assess geometrical abilities and in the same time to help students in developing their skills in argumentation. Silvia Semana examined how the written report, within the context of assessment for learning, helps students in learning geometry and in developing their explanation and argumentation skills at the 8th grade in Portugal. This study suggests that using written reports improves those capabilities and, therefore, the comprehension of geometric concepts and processes. These benefits for learning are enhanced through the implementation of some assessment strategies, namely oral and written feedback.

Anat Levav developed an approach based on the presumption that solving mathematical problems in different ways may serve as a double role tool - didactical and diagnostic. She described a tool for the evaluation of the performance on multiple solution tasks (MST) in geometry. The tool is designed to enable the evaluation of subject's geometry knowledge and creativity as reflected from his solutions for a problem. The example provided for such evaluation is taken from an ongoing large-scale research aimed to examine the effectiveness of MSTs as a didactical tool. Anat Levav argued that this method could be extended to other domains in mathematics.

3D Geometry: Teaching, thinking and learning

The working group was concerned by some studies on 3D Geometry with new viewpoints due to the use of dynamical software in the learning of these specific parts of geometry which is often left aside in the real curriculum. Dynamic Geometry Environments (DGEs) in 2D are one of the well researched topics in mathematics education. DGEs for 3D-environments (Archimedes, Geo3D and Cabri 3D) were designed in Germany and France. Mathias Hattermann studied the specific drag-mode in 3D Geometry environments. He showed that pre-service teachers with previous knowledge in 2D-systems prefer to work with a real model of a cube instead of the 3D-system to solve certain problems. Previous knowledge in 2D-systems seems to be insufficient to handle the drag-mode in an appropriate way in 3D-environments. In a second study, he introduced the students to the special software before the investigation and distinguished different dragging modalities during the solution processes of two tasks.

The approach of Joris Mithalal is more on the transition to formal proof in 3D Geometry. Teaching mathematical proof is a great issue of mathematics education, and geometry is a traditional context for it. Nevertheless, especially in plane geometry, the students often focus on the drawings. As they can see results, they don't need to use neither axiomatic geometry nor formal proof. He tried to analyse how space geometry situations could incite students to use axiomatic geometry. Using Duval's distinctions between iconic and non-iconic visualization, he discussed the potentialities of situations based on a 3D dynamic geometry software.

In the two last papers, the authors focused on the traditional way of teaching and learning 3D Geometry. Edna Gonzalez presented part of the analysis of a Teaching Model for the geometry of solids of an initial Education Plan for elementary school teachers, and its implementation in the University School of Teaching of the Universitat de València in Spain.

In a statistical analysis of the results of 269 students (5^{th} to 9^{th} grade) in Cyprus, Marios Pittalis tried to show that 3D geometry thinking can be described across the following factors: (a) recognition and construction of nets, (b) representation of 3D objects, (c) structuring of 3D arrays of cubes, (d) recognition of 3D shapes' properties, (e) calculation of the volume and the area of solids, and (f) comparison of the properties of 3D shapes. With these factors, he identified four different profiles of students. In the future, it would be useful to make these kinds of studies in various contexts with other theoretical frameworks to validate the conclusions.

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THE NECESSITY OF TWO DIFFERENT TYPES OF APPLICATIONS IN ELEMENTARY GEOMETRY

Boris Girnat

University of Münster, Germany

This article connects the results of an ontological investigation on elementary geometry to normative questions on educational goals of modelling. The main thesis consists in the assumption that there are two different types of applications in geometry and that they both are necessary and not exchangeable by each other: The first one contains simple applications which are paradigmatic examples to learn basic geometrical concepts; the second one includes more complex ones. It is claimed that a normative discussion on education goals of modelling is only possible as far as the second type is concerned. As a result, the debate on modelling differs in the scope of geometry significantly from similar considerations relative to other parts of mathematics, and that by an ontological and not normative reason.

A CASE STUDY TO RETHINK THE ROLE OF APPLICATIONS

This article is a result of a qualitative study concerning teachers' beliefs (Calderhead 1996) about teaching geometry at German higher level secondary schools (the so-called Gymnasien) including goals, contents, methods and connections to the teachers' broader understanding of mathematics as a whole system. The theoretical framework follows the psychological construct of subjective theories which are defined as systems of cognitions containing a rationale which is, at least, implicit (Groeben et al. 1988). The method depends on case studies. Data are collected by semi-structured interviews and interpreted according to the principles of classical hermeneutics. The construct of subjective theories and its adaption to the didactics of mathematics are briefly summed up by Eichler (2006).

In the following, a small part of this study will be presented. We will describe the difficulty of making sense of a teacher's utterances concerning geometrical applications. This difficulty was the initial point to *rethink the role of applications in elementary geometry in general*. Such a way of rethinking is one of the typical goals intended by the construct of subjective theories: This approach proposes, amongst others, to establish an exchange between individual opinions of "practising semi-specialists" and the theories of the scientific community.

A TEACHER'S OPINION ON APPLICATIONS IN GEOMETRY

The teacher of the case study presented here – let us call him Mr. B – has been taught mathematics, physical education, and computer science at a German secondary school for approximately 25 years. The age of his pupils ranges from 10 to 19 years. He seems to be well grounded in mathematics education and equipped with an elaborated concept of school-compatible mathematical applications. As a part of his position, he is involved in the education of trainee teachers in mathematics. This may be a further indication for the assumption that he is familiar with recent theories and perspectives of didactics.

As far as applied mathematics is concerned, his criteria for "good" applications match a lot of the attributes which are discussed and accepted by professional didacts (cf. Jablonka 1999). He demands that "the result [of a model building process] has to be useful for practical acting and reasoning" and that the real-world problems have to be "authentic and realistic, and not artificial and constructed" fulfilling their educational functions by being "challenging, but solvable – possibly after and due to simplification" (all quotations are translated by the author). He mentions the concepts of modelling and model building processes explicitly and approves the new style of arguing which is introduced to mathematics education by mathematization. He concludes: "Modelling and mathematical applications – these are things for which I would never abandon just a minute to discuss an automorphism instead."

AMIDST A STRUGGLE OF TENDENCIES?

At first sight, Mr. B seems to be a true advocate of model building processes and mathematization. But later, when asked how significant applications are for his everyday lessons taught in geometry, he admits that it is "not easy to find good geometrical applications." He refers to some examples taken from computer-aided design, navigation and traffic routing, but – as the main surprise – he does not expect that these applications are the ones his students should keep in mind. They should rather gain "an understanding of spatial relations" and forms and symmetries and they ought to deal with "rather simple applications" like drawing and folding figures or "reading a city map"; and finally, he does not ask which abilities can be conveyed by modelling and mathematization, but, instead, in which cases modelling is "more necessary for the students" – and one can add: to understand geometry.

At this point, there appears to be a rupture, possibly an inconsistency in Mr. B's perspectives concerning geometrical applications. On the one hand, he stresses the abilities and capacities in modelling and problem solving, which could be enforced by using authentic and challenging real-world problems; on the other hand, he regards "simple" geometrical applications as a tool to understand the concepts and theorems of elementary geometry – highlighting the knowledge of geometrical objects, of their attributes and dependencies as an educational goal on its own, and not as a device to manage practical challenges and to build up general skills beyond the scope of mathematics. The parts of goals and means seem to be suddenly switched over.

At first sight, there might be a simple and obvious explanation for Mr. B's ambivalent statements: He could be influenced by two different schools which Kaiser claims to have located within the discussion on mathematical applications (Kaiser 1995). She distinguishes between a *pragmatic* and a *scientific-humanistic approach*: In the pragmatic view, mathematics is a tool to solve practical problems. Applications are deemed as practices to achieve problem solving capacities in managing real-world issues (Kaiser 1995, p. 72). Therefore, applied mathematics is seen from a *procedural* point of view and modelling and model building *processes* are stressed as a content of the curriculum. The scientific-humanistic school, in contrast, emphasizes the principle of “conceptual mathematization”, that means that real-world situations are used to discover and develop mathematical concepts and insights and to receive mathematical ideas based on manifold associations (Kaiser 1995, p. 72).

GEOMETRICAL WORKING SPACES

To clarify the ideas of the scientific-humanistic school as far as geometry is concerned, it is suitable to use the theoretical framework of *geometrical working spaces* (summed up by Houdement 2007). By this approach, geometry is split into three different paradigms (Houdement & Kuzniak 2003):

1) Geometry I (Natural Geometry): Geometry is seen as an *empirical science* which refers to physical objects. To proof or to refute conjectures, both *deduction and experiments* are allowed, whereas measurement is the main experimental technique. This theory is *not axiomatic*, and its type of deduction is similar to inferential arguments between “local ordered” propositions in ordinary language discussions.

2) Geometry II (Natural Axiomatic Geometry): Geometry is treated as an *axiomatic theory*. The axioms are supposed to refer to the real world and, therefore, to describe physical figure and objects (with some idealization). Insofar, Geometry II is *empirical*, too. But to proof or to reject propositions, no empirical argument is permitted, but only a *deductive* one based on the axioms.

3) Geometry III (Formalist Axiomatic Geometry): Geometry is seen as an *axiomatic and deductive theory*, and no connection to the real world is intended.

With reference to this approach, the main goal of the scientific-humanistic school can be described as the project to prevent a sudden transition from Geometry I in primary school to Geometry III in the higher level secondary school in Germany. Such a sudden transition was enforced by the scientific tradition of this type of school and even increased by the New Maths movement until the early 1980s (Schupp 1994).

The alternative drift of the scientific-humanistic school was to fortify Geometry II, to establish a tender segue from Geometry I to II, and finally to achieve Geometry III or, at least, an idealistic interpretation of Geometry II which replaces the reference to physical objects by the platonic idea of *idealistic objects* not being present in the physical world. This project was mainly motivated by two reasons (cf. Kaiser 1995, p. 73): On the one hand, the ontological binding to real-world objects should be an *intermediate* stage on the way to an idealistic or formalist view of geometry to prevent a not understood formalism. On the other hand, it should establish an understanding of the role geometry plays as a tool in natural sciences. In both cases, the ontological foundation in real-world objects was primarily not intended to enforce model building processes and skills, but to build up a “field of associations” in order to understand geometry or natural science more proficiently.

NORMATIVE ISSUES OF APPLIED MATHEMATICS

Concerning applied mathematics, the pragmatic and scientific-humanistic approach differ in weighting *normative parameters*: One of them sets priorities in practical relevance und abilities to deal with model building processes; the other one stresses the theoretical aspects of mathematics (and natural sciences) and uses the associations to real-world situations as a tool to achieve a deep and connected understanding of mathematical concepts. The origin of this controversy appears to be nothing else but a disagreement about educational goals; and the different role of applications does not seem to arise from a specific character of geometry or geometrical applications, but only from disparate normative points of view – a situation which seems to have the same consequences in every part of mathematics and mathematics education, and not only in matters of geometry.

Exactly this opinion is called into question by our following considerations. We will propose an alternative assumption to explain the main statements of Mr. B. Our explanation is based on two arguments: Firstly, we will discuss an investigation on the *ontology* of geometry to clarify the question whether geometrical applications can be treated in the *same way* as other ones. Secondly, we will concern transcendental arguments to elaborate the issue to what extend the use and choice of geometrical applications are within the scope of *normative* deliberations.

THE STRUCTURAL THEORY OF EMPIRICAL SCIENCES

Our ontological consideration is influenced by a particular kind of *philosophy of science* which is called the "structuralist theory of empirical sciences", primarily established by Sneed and later elaborated by Stegmüller and others (Sneed 1979 and Stegmüller 1973/1985). The core assumption of this approach is the idea that empirical theories can be described by two components, namely by a *set-theoretical predicate* and a *set of intended applications* (Stegmüller 1973/1985, pp. 27–42). The set-theoretical predicate contains all of the formal and axiomatic aspects and is defined by the same method used by mathematicians in succession of Bourbaki: In the same manner, how it is possible to define the concept of a group as a pair (G,*) so that every element of G fulfils certain axioms relative to *, the axiomatic background of classical mechanics can be expressed by a quintuplet so that every element of the carrier set fulfils the well-known Newtonian axioms (Stegmüller 1973/1985, pp. 106–119).

At this stage, there is no difference between an empirical and a non-empirical theory (for example a mathematical theory from a formalistic point of view): They both can be defined by set-theoretical predicates. The difference arises from the set of intended applications: In case of non-empirical theories, this set is empty. In case of an empirical theory, it contains the applications which are claimed to be describable and explainable by the concerned theory. For instance, some of the intended applications of classical mechanics are pendulums, solar systems and especially apples falling from a tree. The set of intended applications cannot defined extensionally, but only by enumerating paradigmatic examples and by declaring that every entity also belongs to this set which is "sufficiently similar" to the paradigmatic examples – leaving vague what "sufficiently similar" means (Stegmüller 1973/1985, pp. 207–215).

The concept of geometrical working spaces is a useful framework to establish a connection between geometry and the structuralist theory of science: Geometry I and II are empirical theories insofar they are intended to refer to real-world objects, and they even share the same set of intended applications: physical objects of middle dimension, especially drawing figures and tinkered matters which are used at school. But despite sharing the same set of intended applications, these theories fundamentally differ in their set-theoretical predicates: Whereas Geometry II is assumed to fulfil an axiomatic system of Euclidean Geometry, the propositions of Geometry I may be so vague and psychologically motivated and so variable relative to different times and persons that they certainly cannot be transferred to a system of axioms and accordingly to a defining set-theoretical predicate. In contrast, Geometry III is not an empirical theory, since it is regarded in a formalist manner, presupposing not to have any applications; that means, in this case the set of intended application is empty. But on the other hand, Geometry III shares the same defining set-theoretical predicate with Geometry II: They both are intended to be a Euclidean Geometry.

The set of intended applications is not just an "illustration", a nice, but useless thing which can be left out; it rather fulfils two indispensable functions: From a logical point of view, the set of intended applications is a conceptual attribute and a part of the *definition of an empirical theory*. It distinguishes an empirical theory from a non-empirical one und declares the "part of the world" to which the theory is connected. Exactly this is the difference between Geometry II and III.

The second function results from the fact that every non-trivial empirical theory is based on idealization. For example, classical mechanics presupposes the existence of point particles without any spatial dimension. However, such entities do not exist in a strict sense of the word, but only "approximately" – and this is the second task of the set of intended applications: Since there is no way to explain explicitly under which condition and to what extent an approximation is allowed to make an empirical theory applicable (Stegmüller 1973/1985, pp. 207–215), i. e. under which condition an application belongs to the set of intended application, the paradigmatic examples of this set provides a number of "case studies" by which the *limits of approximation* are implicitly defined and novices of the scientific community can become familiar with the scope and borders of their coming occupation.

In geometry, the problem of approximation will typically arise, if infinity or dimension zero occurs; straight lines, planes, and angles are paradigmatic examples of this case (Struve 1990, p. 43). For instance, if there is a line drawn on a paper, there will be two ways to deal with the question "Is this a straight line, a segment of a straight line or neither of them?": From a formalist or idealistic view of geometry, this is a trivial question, since geometry does not refer to physical objects; a physical line is neither a segment nor straight line; at most, drawings could be symbolic tools to think about geometrical objects or propositions. But if it is taken serious that geometry can be interpreted as an empirical theory (as supposed in Geometry I and II and as being common and necessary for geometrical applications as we will see later), the pupils will have to learn to treat a line sometimes as a segment and sometimes as a straight line. To deal with these decisions is a notorious problem in geometry. The intended applications like drawing figures are the paradigmatic examples by which pupils are supposed to learn to manage these questions.

Hence, the knowledge of the set of indented applications and the handling of its vagueness is not optional, but an integral part of a particular empirical theory and, therefore, one of the aspects of "possessing" and being able to apply a certain theory. The educational task of paradigmatic examples is primarily described by Kuhn as far as philosophy of science is concerned (Kuhn 1962/1976, pp. 59–62). It is also a common thesis in psychology that paradigmatic examples play a major role in learning a theory (e. g. Seiler 2001, pp. 144–225).

ONTOLOGICAL ASPECTS OF ELEMENTARY GEOMETRY AT SCHOOL

At this point, we will come back to didactics. Struve has investigated how elementary geometry is presented in secondary school following the philosophy of science structuralism sketched above (Struve 1990, p. 6). Expressed in terms of the theory of geometrical working spaces, he comes to the conclusion that the didactical changes which were established to avoid a sudden switch from Geometry I to Geometry III by stressing Geometry II (as mentioned above) factually took the effect that the new textbooks present rather Geometry I than Geometry II and (even if Geometry II is reached) geometry is continuously taught as an empirical theory, and never as a formalistic or idealistic one as intended: "students learn an empirical theory in the geometry lessons held at secondary school" and "concerning the empirical theory, as we want to call the theory the students learn in their geometry lessons according to our investigation, figures created by folding and drawing are the paradigmatic examples" (Struve 1990, pp. 38–39).

THE ISSUE OF MODELLING

Struve has mentioned some of the consequences of his result – foremost some consideration on the fact that proofs have different functions in empirical and non-empirical theories observing that students typically treat proofs in the same manner as they are used in empirical sciences (Struve 1990, pp. 38–49). In this article, we will add a consideration concerning modelling. If we can follow Struve's results, Mr. B's distinction between two types of geometrical applications is not confusing, but an obvious implication of the empirical character of geometry as it is taught in secondary school: The figures created by drawing and folding and the "simple" applications based on these figures can be regarded as the paradigmatic examples which define the set of intended applications and constitute geometry as the empirical science of the spatial environment surrounding us in everyday life.

In this view, the supremacy of simple applications is not based on a normative decision about the role of application in mathematics education, but on the *specific ontology of geometry*: The knowledge of and the familiarity to these examples of applications are defining attributes of geometry as an empirical science. Hence, with regard to these "basic" applications, geometry differs from the other parts of mathematics taught at school. In the other cases, the amount and choice of applications is a normative question guided by arguments which Kaiser has combed through. In geometry, however, the task of normative deliberations begins not before the set of intended applications is left. Therefore, it is not astonishing that the (rare) cases which Mr. B mentions as "real" examples of modelling in geometry are quite different from the paradigmatic examples of folding and drawing: computer-aided design, navigation and traffic routing. In these cases and after some basic courses based on "simple" applications, geometry may no longer differ in modelling and mathematization.

TRANSCENDENTAL ASPECTS OF GEOMETRY

Our last task concerns the question if the dominance of an empirical view of geometry at school (as Geometry I or II) is an aberration caused by psychological circumstances and enforced by "misguided" textbooks or if there are good reasons to teach geometry as an empirical theory (to some extend). We will argue for the latter, accentuating a special role of geometry in contrast to other parts of mathematics and aiming for the conclusion that therefore two different types of applications are needed.

Let us start with an example: In 2003, a new national curriculum framework called "Bildungsstandards" (educational standards) was established in Germany. In contrast to former resolutions, this declaration stresses general skills, abilities and competencies – and among others, abilities in mathematical modelling. The relevant paragraph closes with the following sentence: "This includes translating the situation which is to be modelled into mathematical concepts, structures and relations" (KMK 2004, p. 8). This is a formulation ranging over all parts of mathematics taught at secondary school. A specific statement focussing on geometry is not declared.

Let us deliberate what this sentence presupposes: There is a real-world situation which can be described by mathematical concepts, but need not to be treated in this way. For instance, you can cross the road without thinking about the probability to be knocked over and you can look at the carps in a lake without having a function in mind to describe their growth process. Normally, a mathematical description is *not necessary* and will only be introduced, if it promises deeper insights as a description in ordinary language. Besides the general skills, this is a typical educational goal of modelling: the awareness that mathematics is a useful tool to achieve knowledge of the external world and to formulate this knowledge in a very precise manner.

In geometry, the case is quite different. If geometry could be treated like other mathematical theories, it would be possible to describe a situation geometrically *only on demand*. But this assumption fails since it is inevitable to use, at least, rudimental geometrical concepts to describe a situation at all. You cannot cross the road or look at the carps in the lake without possessing, at least, a broad understanding of basic geometrical concepts. For instance, a (vague) understanding of relative positions is necessary to individuate the different things, persons or objects which are part of a specific situation.

The idea that space is not a thing of human perception among others, but the conceptual framework which allows to describe real-world phenomena was primarily introduced by Kant as a part of his *transcendental* philosophy (Kant 1781/1998). In contemporary ontology the conceptual framework of space (and time) is broadly accepted as a condition to describe real-world situations (for everyday perceptions see Runggaldier and Kanzian 1998, pp. 17–52, as a condition of empirical sciences see Bartels 1996, pp. 23–71, or Stegmüller 1973/85, p. 60).

CONCLUSION: TWO TYPES OF GEOMETRICAL APPLICATIONS

Now, it is possible to connect both arguments: Following transcendental considerations, it is necessary to possess basic concepts to describe real-world situation and to establish the conditions under which model building processes are possible. That means, for mathematical reasons it may be passable to interpret geometry as a formalist or idealistic theory; but for model building processes or in contexts of natural sciences, it is necessary to understand geometry as an empirical theory. For some simple model building processes, an understanding on the level of Geometry I may be sufficient, but for more elaborated tasks or as a tool of natural sciences, Geometry II seems to be indispensable.

Against this background, we attain a “two step view” of geometrical applications: Since concepts of an empirical geometry are necessary to apply mathematics and, in a structuralist view of science, these concepts correspond to a set of intended applications taken from the world of folding and drawing, the first type of applications consists of very “simple” applications whose function is completely defined by learning and applying elementary geometry, especially by learning to manage the reference of concepts like “straight line” which can only be applied due to approximation. Hence, geometrical applications of a “simple” kind are *inevitable ingredients* of teaching geometry; and there is no reason to criticize the simplicity of these applications. At this stage, a normative debate about goals of teaching “applied geometry” is inadequate, since according to the empirical character of school geometry, there is no difference between teaching applied geometry and teaching geometry at all. This shall be our first conclusion: To some extend, it is necessary to deal with simple geometrical applications; and this necessity is not an inference from a normative decision about the goals of teaching applied mathematics, but a consequence of the specific ontological situation of geometry and it transcendental function as a condition of natural science and ordinary perception. No other part of secondary school mathematics possesses this ontological and transcendental function. For this reason, the status of geometry is unique, and the debate on geometrical applications cannot be held in the same way as it is possible in the scope of other parts of mathematics.

The second conclusion is related to the other type of geometrical applications: If the “simple” and intended applications are the only ones which students get to know, there will be an obvious deficit in teaching general skills and model building capacities in the sense of the pragmatic view of applied mathematics. Exactly this is the function of the second type of geometrical applications. It is comprehensible that applications which are intended to fulfil this task are quite different from the first ones. Mr. B mentions examples taken from computer-aided design, navigation and traffic routing. A list of similar examples is collected by Graumann (1994). Applications of this kind are typically not “pure geometrical”, but includes concepts or hypotheses taken from natural or social sciences, basic economics or empirical tedium platitudes. This fact can be regarded as a further indication for our claim that there two different types of applications with distinct functions: Whereas the simple ones are used to built up geometrical concepts and to manage the vagueness of applying geometrical concepts to real-world situations, the more complex ones are intended to use *pre-existing* geometrical concepts and insights to reach some of the many educational goals which Kaiser sums up for model building processes in general (Kaiser 1995). For this purpose, a real-world problem only providing geometrical aspects often does not appear to be multifarious enough to allow a model building process whose challenges lie in this process (including mathematization, simplification, validation and hypothesis testing), and not in geometrical deliberations and calculations.

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A FRENCH LOOK ON THE GREEK GEOMETRICAL WORKING SPACE AT SECONDARY SCHOOL LEVEL

Alain Kuzniak and Laurent Vivier

Equipe Didirem Université Paris-Diderot, France

Based on the geometrical paradigms approach, various studies have shown some tension in French Geometrical Working Space between institutional expectation and effective implementation. In this paper, we examine the Greek system from this point of view and we find the same kind of tension but in a certain sense stronger than in France even if both countries have an ancient Euclidean tradition.

FROM SPECIFIC FRENCH CASE TO THE PARTICULAR GREEK CASE

Since several years, it seems that curricula and syllabi converge to promote a close link between mathematics teaching and the "real world". The idea of "mathematical literacy" is especially strong in the PISA evaluation which aims to organize this general trend among European countries. At the same time and close to this conception of mathematics, the constructivist approach is favoured by national educational institutions and teachers are asked to substitute "bottom up" teaching methods to the traditional "top down" entrance in mathematics.

In France, till today, and at lower secondary school level the prominent way suggested by the intended curriculum is based on "inquiry methods" and "activities" and relationships between mathematics and other scientific or technological domains are always pointed up. But the link to sensible world is only mentioned and the emphasis is put on the logical rigour of mathematics. The relationship to the "real world" seems really far off and into everyday classroom, inquiry based methods are left aside.

In the special case of geometry, we were concerned with the contradiction between official expectation and the crude reality of the classroom. To understand and explain the phenomenon, the notion of geometrical paradigms (Houdement and Kuzniak, 1999) and of geometrical working spaces (Kuzniak, 2007) have been used to explicit the different meanings of the term geometry. The field of geometry can be mapped out according to three paradigms, two of which – Geometry I and II – play an important role in today's secondary education. Each paradigm is global and coherent enough to define and structure geometry as a discipline and to set up respective working spaces suitable to solve a wide class of problems.

This first idea is completed by the following hypothesis on the possible influence of these paradigms in geometry education and on the poor implementation of new teaching method. The spontaneous geometrical epistemology of teachers enters in contradiction with mathematical epistemology embedded in the new teaching methods. In other words: the geometrical work done and aimed by teachers could be of another nature than the institutional expected one. The teacher’s geometrical thinking is led by another geometrical paradigm as the paradigm promoted by the institution. Moreover this way of thinking leads to prefer pedagogical methods in contradiction with inquiry based methods.

Our investigation work has its roots in the French context but some comparative studies showed us that such a tension could exist in other countries. Houdement (2007) has presented in CERME 5 a comparison of magnitude measurement problems in Chile and in France. The social and economical contexts are quite different in both countries and so, we were interested to have a look on other European countries to verify if this kind of tension really exists and how it was managed. We have had the opportunity to work with Greek colleagues and to be aware of a great change in the curriculum based on the real world and turning back to the Euclidean tradition. We present the first part of our work which gives our analysis of the Greek situation through our viewpoint.

GENERAL FRAME OF THE STUDY

The theoretical frame we used has been soon described in detail in former CERME sessions (Houdement and Kuzniak 2003, Houdement 2007) and we refer to these papers for complements. We retain only here some particular elements used in our description of the Greek situation.

As we are interested in the awkward relationships between reality and mathematics education, we will focus on the role the reality plays in the different paradigms. In the first one, Natural Geometry or Geometry I (GI), the validation depends on reality and the sensible world. In this Geometry, an assertion is accepted as valid using arguments based upon experiment and deduction. The confusion between the model and reality is great and any argument is allowed to justify an assertion and convince. This Geometry could be seen as an empirical science and it is possible to build empirical concepts depending on the experience of the “real world”. Natural Axiomatic Geometry, or Geometry II (GII), whose archetype is classic Euclidean Geometry is built on a model that approaches reality. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. In the formal Axiomatic Geometry, or Geometry III (GIII), the system of axioms, which is disconnected from reality, is central and leads how to argue. The system of axioms is complete and unconcerned with any possible applications in the world. In that case, the system creates its reality. Concepts are given *a priori* and come “from the Book” and so “top down” form of mathematics education seems well fitted to this conception. The study of Greek mathematical education will show that this dichotomy GII / GIII is not so simple.

To find a possible tension or contradiction between the institutional expectation and the teacher's approaches, we will describe what we call the personal teacher's Geometrical Working Space (GWS) faced to the GWS expected and promoted by the national institution in charge of mathematics education. More precisely (Kuzniak

2006), the Geometrical Working Space (GWS) is the place organized to ensure the geometrical work. It makes networking the three following components: the real and local space as material support, the artefacts as drawings tools and computers put in the service of the geometrician and a theoretical system of reference possibly organized in a theoretical model depending on the geometrical paradigm. To ensure that the components are well used, we need to focus on some cognitive processes involved into the geometrical activity and particularly the visualization process with regard to space representation and the material support, the construction process depending on the used tools (rulers, compass, etc.) and on the configuration, and finally reasoning in relation to a discursive process.

THE NEW CURRICULUM IN GREECE

Since 2007, a new curriculum for compulsory education is implemented in gymnasium (grades 7 to 9) in Greece and summarised in a list of ten highlights. It is presented as cross-thematic (1st and 5th highlights) and aims to connect the academic disciplines, everyday life, working world, history, technological improvement, etc. Within the flexible zone (4th highlight), some hours are planned for reaching this specific goal. Primary school learning explicitly rests on the Bruner's constructivist theory and assessment is now an essential part of the learning process (8th highlight). Sources and goals of connection with realty are in the 9th highlight, “A Broad Spectrum of Literacies”:

> Successful living in post-modern times presupposes that one is fully literate in many areas, such as reading, science, technology and mathematics in order to face international evaluation (PISA, TIMS, etc.) which demand more connections between school knowledge and the life reality.

The present mathematical syllabus expands the ancient one with no change in the content. It is written in a three columns table where some more detailed mathematical sections appear into the traditional blocks (arithmetic, algebra, geometry). Mathematical skills, which have to be learned by pupils, are described in the first column, the main mathematical notions are in the second and in the third one some activities are proposed, often to introduce some mathematical notions.

New textbooks are conformed to syllabus with no surprise since they are chosen by the curriculum designer Pedagogical Institute, one for each level. Textbooks structure is quite the same as the syllabus structure and activities coming from the syllabus third column can be found with few changes in textbooks. For these reasons, institutional GWS means the GWS presented by the curriculum including the official textbooks.

A SO FAR REALITY

We will highlight some internal slides into the institutional GWS itself. First, in spite of the curriculum demand, new technologies have to be used (7th highlight), syllabus and textbooks do not mention software, computers or Internet. Beside this slide inside the curriculum, the reality is concerned by a second and less obvious one.

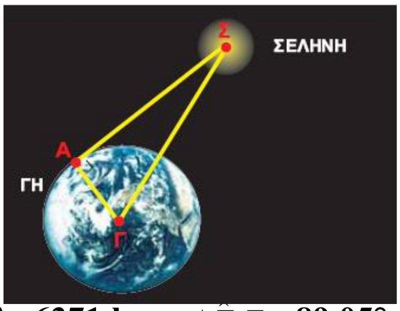
According to the cross-thematic curriculum, reality and everyday life have to be embedded in the learning process. But when everyday life is mentioned in syllabus it is without any details and only one syllabus activity could be described as real : *measure the width of the street and pavement in front of the school*. But the difficulty to follow this curriculum directive is more obvious in textbooks. This real activity in syllabus does not appear in the A' textbook (grade 7), and if there are numerous activities based on a "real picture", they are not relevant for this purpose for several reasons:

- The 3D/2D problem: angles and distances on the textbook are not the good ones. For these kind of activities, geometry does not seem to be able to give the right answer!

Students have to find the lower distance between the point A at the house to the water. (A' page 184)

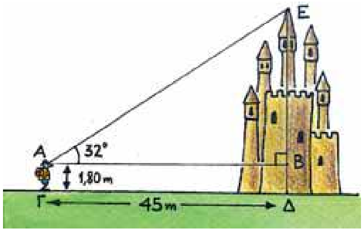
- A lot of activities refer to the macro-space but authors represent reality – probably under editorial constraint – with an image or photography. On these pictures, most of the time, some geometric element are placed and the reality is already mathematized. However, we often find activities and exercises with geographic maps, as it is stated in syllabus. But reality is once more already mathematized.

Why an airplane realize a lower distance than a boat to go from Athens to Samos? (A' page 164)

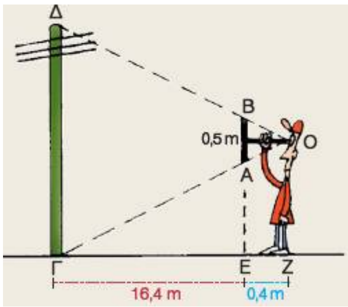


ΓΑ=6371 km, $A\hat{\Gamma}\Sigma$ =89,05°, find ΓΣ. (B' page 151)

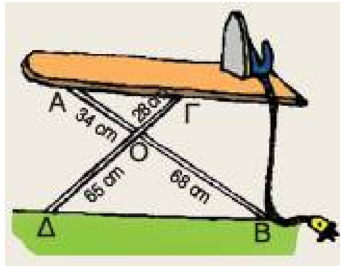
- Activities and exercises are most of the time based on a picture of a real problem with a geometric diagram with all the measures needed to solve the problem, no more no less. Reality is not the point and is viewed through a picture already turned into a geometrical task support.



Find ΔE. (B' page 139)



Find ΔΓ. (G' page 223)



Why it is not horizontal? (G' page 209)

As we notice it, the geometrical local space is almost always the micro-space of a sheet of paper which is sometimes a representation of a macro-space problem (geographic maps, pictures, etc.). Actually, the reality in textbooks appears from a relevant point of view only in the GI paradigm [1], on a sheet of paper. And so we can characterize this internal slide: everyday life is not taking into account and reality is only treated within the GI paradigm, inside geometry.

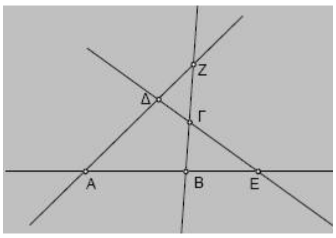
GYMNASIUM INSTITUTIONAL GWS

Since reality is not actually present in institutional GWS, except within the GI paradigm, we study the institutional GWS all along the gymnasium.

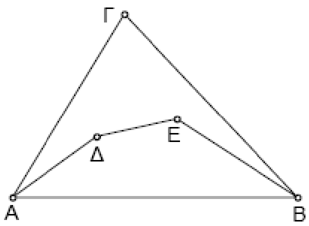
Artefacts, visualization and diagrams constructions: the GI paradigm

Geometric tools (ruler, compass, protractor, square, tracing paper) are only mentioned in syllabus at the A' class (grade 7). However, construction activities are present all along the gymnasium (much more at the first class). In the A' textbook, tools are pictured in many places, especially for showing how to construct. Tracing paper is used in many geometry sections, often to introduce a new concept. In the B' and G' textbooks (grades 8, 9) geometric tools are never drawn, sometimes mentioned.

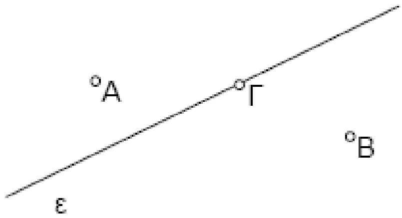
There is no freehand construction in syllabus, no freehand diagram in textbooks and we do not find any exercise where pupils have to draw such a kind of diagram. Some activities proposed in syllabus (third column of A' class) are in GI, excluding, or not, visualization:



How many angles?



Find in measuring the lower distance between A



Draw the perpendiculars to ε passing by these points.

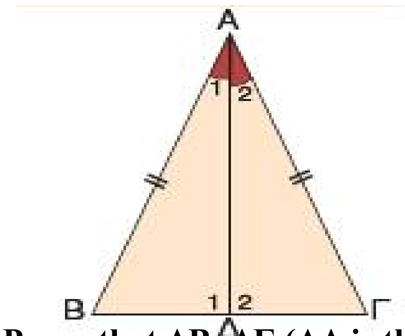
An aim of syllabus, at B' class (grade 8), section trigonometry, is to construct an angle whose sinus, cosine or tangent are known. But we do not find any activity on this topic in textbook. At the final class (grade 9) the section on dilation is directed by the GI paradigm with numerous drawing activities (7 exercises of the 9 at the end of the section ask for drawing).

Formal proofs: the GII paradigm

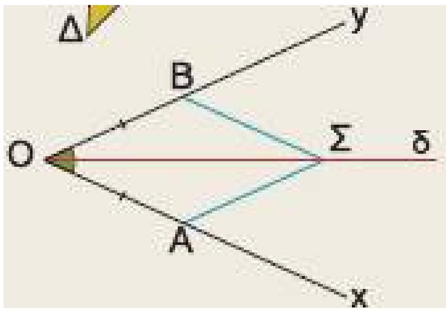
Proof process should start as it is written in syllabus preamble, but no formal proof is mentioned in the detailed table of mathematics syllabus. There are some theorems, definitions, properties.

Very few examples of formal proofs are given in the A' textbook (grade 7) and their solutions are always completely written. It is quite the same situation in the B' textbook (grade 8), except the proof that a dodecagon is regular (exercise 8, page 185). In the B' area section, a lot of exercises ask to "show that" but, in fact, the solution is always given by a calculation of an area or a length.

In G' textbook (grade 9) there is a great change with a lot of exercises where pupils have to prove. At the section on triangle congruence, the 21 exercises at the end of the section ask for a formal proof and the theoretic system of reference, with the three criteria of triangle congruence, is clearly directed by the GII paradigm. In this section, there are four solved exercises (pages 191, 192) which ask for a formal proof on triangle congruence (see below, for example, the figure on the left). At the end of the section (pages 194-196) some similarly exercises are given (see below, for example, the figure on the right). One could thought that the solutions of the four solved exercises could give a proof model to students to solve exercises at the section end.



Prove that ΔB=ΔΓ (AΔ is the bisector of Â). <u>With solution</u>. (G' page 191)



Prove that AΣ=BΣ (OA=OB, Oδ is bisector of Ô). Exercise <u>without solution</u>. (G' page 194)

The diagrams similarity section is also in GII paradigm (half of the exercises ask for a formal proof, the others are on ratio and length calculation).

Gymnasium paradigm

At the first class A', both in curriculum and textbook, the main paradigm is GI and it is generally well assumed. However, the paradigm in which pupils have to work is not always clear. For example, the following syllabus activity starts in GI and finishes, with questions g) and h), necessarily in GII:

a) Let O a point and a line ε and the point A so that OA is the distance from O to ε.

b) Let B another point on ε, find the symmetrics A' and B' of A and B through O and let ε' the line A'B'.

c) Which is the symmetric of ε through O ?

d) Which is the symmetric of the angle OÂB ?

e) How are the angles OÂB and OÂ'B'?

f) How is the angle OÂ'B'?

g) How are ε and ε' with respect to AA'?

h) How are ε and ε'?

Didactic contract is not very clear for the intermediate questions c), d) and e): GI, with tools or visualization, or GII paradigm? This activity is given in textbook with only one question and a complete solution below. The task paradigm is clearly GII: the answers corresponding to questions e) to h) are formal proofs. This example is a non explicit slide from GI to GII in a class where GI is the main paradigm [2].

Artefacts and diagrams constructions are used in many activities to discover geometrical properties, as it is written in the curriculum according to the bottom-up point of view: from the GI paradigm arises the GII paradigms. Some activities given in the third column of syllabus are in GI, to construct, to observe a property (sometimes in first class with the use of tracing paper and folding). This kind of activities can be find in all gymnasium textbooks (grades 7 to 9).

In gymnasium, from grade 7 to 9, geometrical tasks are very different. The GWS depends on the class and the section. In the first class GWS is clearly directed by GI but there are some slides in favour of the GII paradigm. In the last class, the GWS of the triangle congruence section is directed by GII while it is directed by GI in the section on dilation. In this last class, there are several very different GWS which seem not to be connected.

EUCLIDEAN PRESSURE ON TEACHER’S PERSONAL GWS

This section is supported by six secondary teachers’ interviews where we focussed on the new curriculum and more specifically on reality, geometrical tools, diagram constructions and formal proofs in textbooks and in classrooms. We turn out to teacher’s personal GWS which is quite different from the institutional one as we will show it. Before studying the GWS teachers, we point out the particular importance of Euclidean Geometry in the Greek syllabus and for Greek teachers.

The paradoxical place of Euclidean geometry

According to the Lyceum syllabus, students have to learn a geometry based on axioms with formal reasoning (grade 10) and measurement of magnitudes becomes the main geometric topic at grade 11. The unique geometry textbook is entitled “Euclidean Geometry” and it is used in the two first classes (grade 10 and 11). Its content is close to the syllabus and to the classical Euclidean Geometry with a strong axiomatic point of view, except for measurement. In textbook, and for lyceum teachers, geometry starts from zero with Euclidean axioms. Construction problems are of theoretical nature with letters and magnitude, such as AB=*a*, without any measure: geometrical tools are virtual and consist of compass and ruler according to the Euclidean tradition.

If Geometry is taught in compulsory education and during the two first lyceum classes (till grade 11), geometric knowledge is not assessed at the very important lyceum final test: the University where students will enter depends of this final test. Students know this fact and are less concerned with geometry than the others mathematics domains and do not work geometry especially in the numerous private institutes (*frontystiria*) where they could follow additional and expensive courses after the class time. It is a quite great contrast: a lot of geometry teaching times for nothing at the end? Teachers we interviewed told us that geometry is not important in the curriculum because of the hidden curriculum and, finally, "geometry is taught for culture, for Euclid".

Teachers' personal GWS

Gymnasium teachers think that pupils have to learn how to construct geometric diagrams, but they think that it is not the main point of mathematics learned in gymnasium. So as they have no time to teach all the syllabus, teachers often choose to teach very quickly diagrams constructions despite its importance and the fact that students have troubles with the use of drawing tools (especially the protractor) and with constructions. In the personal teacher's GWS, directed by GII, the aim of a diagram is to set a conjecture and the proof do not need an exact figure. That explains why teachers think that a freehand drawing is equivalent to a drawing with geometrical tools and the first one is done more quickly. Teachers' local space could be anywhere they can draw a freehand diagram, for example a pack of cigarettes as two teachers told us. We see here a great difference between teachers' beliefs and institutional content: in syllabus, nor in textbooks, there is none freehand drawing.

Another example of the prominence of GII in the personal teacher GWS is the importance they give to properties of quadrilaterals and triangles. They all think that these properties are fundamental even if they do not know the role of these geometric objects in mathematics class. As teachers rate highly Euclidean Geometry, a sufficient reason to teach triangles and quadrilaterals is given by their importance in the theoretic system of reference.

To conclude this part, we can say that the teachers' GWS is clearly directed by a strong GII, almost GIII because of the axiomatic theoretic system of reference.

GWS TENSION

The new Greek curriculum demands to take into account reality. But the interviewed teachers told us how it is difficult for them: they do not know how to teach in a constructive way which is often opposite to their top-down learning conception. They concluded that Greek teachers do not like this new way of teaching and do not understand it. Teachers' learning beliefs agree with the internal slide we pointed out about the everyday life in curriculum.

In the case of diagrams constructions, teachers' GWS is clearly against the institutional GWS, and not only in considering freehand drawing. Teachers do not only prefer teaching others geometric topics but they give all the diagrams in tests too to go over the lack of their pupils [3]. The same opposition to the institutional GWS can be seen with the use of tracing paper. According to syllabus, tracing paper has to be used as a geometric tool in A' class (grade 7). It is used in many places with a particular and original graphical representation in the A' textbook and it is explained how to use it. But creativity stops at the school border and tracing paper is never used in class!

In gymnasium, formal proof is usually taught during the last class year (grade 9), more specifically, in a Euclidean section about triangle congruence. In order to know how teachers could initiate their students to the formal proof in one year, we asked them about the possible use of the four solved activities we spoke about in the "Formal proofs: the GII paradigm" section. They are indeed proof models and, for assessment, students have to learn ten lesson proofs by heart which one of them is asked in test. This proof process initiation is again opposite to the curriculum expectation.

In gymnasium, there is a distance between institutional and teachers GWS. That creates a tension which is supported by the different beliefs on learning and geometry among teachers and curriculum writers. Moreover, teachers do not really deal with the existing and remaining students' difficulties with diagram constructions and the proof process initiation is based on a learning by heart. This tension between institutional and teachers' GWS is specific to gymnasium, it completely disappears at lyceum, but what about pupils?

CONCLUSION

Geometry positions in Greece and in France are closed even if we point out some main differences. In both countries, even if curriculum emphasizes its place, reality is not taken into account. Similarly, the transitions between paradigms GI and GII are most of the times ambiguous and implicit and give rise to fuzzy GWS.

The GI paradigm seems to be more assumed in Greece than in France and in France formal proofs are taught all along the junior high school. But the main curriculum difference takes place at the lyceum: in Greece, axiomatic Euclidean geometry is taught, not in France, and in France geometry is assessed in final test for some sections, not in Greece. Geometry is taught in Greece only for cultural reasons, for Euclid, whereas in France the geometrical work is oriented by the GII paradigm and university studies. However, according to the six teachers' interviews, the Greek teachers' GWS is quite different from the French teachers' GWS because of the axiomatic theoretic system of reference: GII paradigm is well structured and stronger in Greece than in France. In Greece, the cultural tradition of Euclid is more important than in France and geometry knowledge seems to come from the Book [4]. This last point strengthens the GWS tension in junior high school which seems to be stronger in Greece than in France.

NOTE

1. The exercise on a map are in GI, but it could be solved by visualization or measurement, pupils have to choose.

2. This non explicit slide can also be seen, for example, at page 227 of A' textbook, examples 1 and 2.

3. In the A' final test we studied there is no construction; lyceum pupils have problems with geometric diagrams constructions, even for the equilateral triangle whereas it is a skill of the A' gymnasium class (grade 7).

4. According to Toumasis (1990) the Book is not Euclid's Elements but Legendre's geometry elements.

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A THEORETICAL MODEL OF STUDENTS' GEOMETRICAL FIGURE UNDERSTANDING

Eleni Deliyianni*, Iliada Elia *, Athanasios Gagatsis*,

Annita Monoyiou* and Areti Panaoura **

*University of Cyprus, **Frederick University

This study investigated the role of various aspects of apprehension, i.e., perceptual, operative and discursive apprehension, in geometrical figure understanding. Data were collected from 1086 primary and secondary school students. Structural equation modelling affirmed the existence of six first-order factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concepts, three second-order factors indicating the differential effects of the various aspects of geometrical figure apprehension and a third-order factor representing the geometrical figure understanding. It also provided support for the invariance of this structure across the two age groups. However, findings revealed differences between primary and secondary school students' performance and in the way they behaved during the solution of the tasks.

INTRODUCTION AND THEORETICAL FRAMEWORK

In geometry three registers are used: the register of natural language, the register of symbolic language and the figurative register. In fact, a figure constitutes the external and iconical representation of a concept or a situation in geometry. It belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws. As a representation, it becomes more economically perceptible compared to the corresponding verbal one because in a figure various relations of an object with other objects are depicted. However, the simultaneous mobilization of multiple relationships makes the distinction between what is given and what is required difficult. At the same time, the visual reinforcement of intuition can be so strong that it may narrow the concept image (Mesquita, 1998). Geometrical figures are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. A geometrical figure is also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). Therefore the double status of external representation in geometry often causes difficulties to students when dealing with geometrical

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problems due to the interactions between concepts and images in geometrical reasoning (e.g. Mesquita, 1998).

Duval (1995, 1999) distinguishes four apprehensions for a “geometrical figure”: perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, *perceptual apprehension* refers to the recognition of a shape in a plane or in depth. In fact, one’s perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. *Sequential apprehension* is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. *Discursive apprehension* is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). However, it is through *operative apprehension* that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refer to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one made the figure larger or narrower, or slant, while the place way refer to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations. These operations constitute a specific figural processing which provides figures with a heuristic function. In a problem of geometry, one or more of these operations can highlight a figural modification that gives an insight to the solution of a problem.

Even though previous research studies investigated extensively the role of external representations in geometry (e.g. Duval, 1998; Kurina, 2003), the cognitive processes underline the four apprehensions for a “geometrical figure” proposed by Duval (1995, 1999) have not empirically verified yet. Keeping in mind the transition problem from one educational level to another universally (Mullins & Irvin, 2000), our main aim was to confirm a three-order theoretical model concerning the primary and secondary school students’ geometrical figure understanding.

HYPOTHESES AND METHOD

In the present paper four hypotheses were examined: (a) Perceptual, discursive and operative apprehension influence primary and secondary students’ geometrical figure understanding, (b) There are similarities between primary and secondary school students in regard with the structure of their geometrical figure understanding, (c)

Differences exist in the geometrical figure understanding performance of primary and secondary school students and (d) Differences exist in the way primary and secondary school students behave during the solution of the perceptual, discursive and operative apprehension tasks. It should be mentioned that the influence of sequential apprehension in geometrical figure understanding is not investigated since the figure construction is not given much emphasis in the Cypriot curriculum.

The study was conducted among 1086 students, aged 10 to 14, of elementary (Grade 5 and 6) and secondary (Grade 7 and 8) schools in Cyprus (250 in Grade 5, 278 in Grade 6, 230 in Grade 7, 328 in Grade 8). The a priori analysis of the test that was constructed in order to examine the hypotheses of this study is the following:

1. The first group of tasks includes task 1 (Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g) and 2 (Pe2a, Pe2b, Pe2c, Pe2d, Pe2e, Pe2f) concerning students' geometrical figure perceptual ability and their recognition ability, respectively.
2. The second group of tasks includes area and perimeter measurement tasks, namely task 3 (Op3), 4 (Op4), 5 (Op5) and 6 (Op6a, Op6b, Op6c). These tasks examine students' operative apprehension of a geometrical figure. The tasks 3, 4 and 5 require a reconfiguration of a given figure, while task 6 demands the place way of modifying two given figures in a new one in order to be solved.
3. The third group of tasks includes the verbal problems 7 (Ve7), 8 (Ve8), 9 (Ve9), 10 (Ve10) and 11 (Ve11) that correspond to discursive figure apprehension. On the one hand, the verbal problems 7 and 8 demand increased perceptual ability of geometrical figure relations and basic geometrical reasoning. On the other hand, tasks 9, 10 and 11 are verbal area and perimeter measurement problems. In verbal problem 9 visualization (e.g. Presmeg, 2007) facilitates its solution process, while in verbal problems 10 and 11 the concept of epistemological obstacles (Brousseau, 1997) may interfere the way of solving them.

Representative samples of the tasks used in the test appear in the Appendix. Right and wrong or no answers to the tasks were scored as 1 and 0, respectively. The results concerning students' answers to the tasks were codified with Pe, Op and Ve corresponding to perceptual, operative and verbal problem tasks, respectively, followed by the number indicating the exercise number.

In order to explore the structure of the various geometrical figure understanding dimensions a third-order confirmatory factor analysis (CFA) model for the total sample was designed and verified. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI and RMSEA. The following values of the three indices are needed to hold true for supporting an adequate fit of the model: x^2/df < 2, CFI > 0.9, RMSEA < 0.06. The a priori model hypothesized that the variables of all the measurements would be explained by a specific number of factors and each item would have a nonzero loading on the factor it was supposed to measure. The model was tested under the constraint that the error variances of some pair of scores associated with the same factor would have to be equal. A multivariate analysis of variance (MANOVA) was also performed to examine if there were statistically significant differences between primary and secondary school students concerning their understanding in the various geometrical figure dimensions. For the analysis of the collected data the similarity statistical method (Lerman, 1981) was conducted using the statistical software C.H.I.C. (Bodin, Coutourier, & Gras, 2000). A similarity diagram of primary and secondary school students' responses at each task or problem of the test was constructed. The similarity diagram allows for the arrangement of the tasks into groups according to the homogeneity by which they were handled by the students.

RESULTS

Confirmatory factor analysis model. Figure 1 presents the results of the elaborated model, which fitted the data reasonably well [$x^2(220) = 436.86$, CFI = 0.99, RMSEA =0.03, 90%, confidence interval for RMSEA 0.026-0.034]. The first, second and third coefficients of each factor stand for the application of the model in the whole sample (Grade 5 to 8), primary (Grade 5 and 6) and secondary (Grade 7 and 8) school students, respectively. The errors of variables are omitted.

The third-order model which is considered appropriate for interpreting geometrical figure understanding, involves six first-order factors, three second-order factors and one third-order factor. The three second-order factors that correspond to the geometrical figure perceptual (PEA), operative (OPA) and discursive (DIA) apprehension, respectively, are regressed on a third-order factor that stands for the geometrical figure understanding (GFU). Therefore, it is suggested that the type of geometric figure apprehension does have an effect on geometrical figure understanding, verifying our first hypothesis. On the second-order factor that stands for perceptual apprehension the first-order factors F1 and F2 are regressed. The first-order factor F1 refers to the perceptual tasks, while the first-order factor F2 to the recognition tasks. Thus, the findings reveal that perceptual and recognition abilities have a differential effect on geometrical figure perceptual apprehension. On the second-order factor that corresponds to operative apprehension the first-order factors F3 and F4 are regressed. The first-order factor F3 consists of the tasks which require a reconfiguration of a given figure, while the tasks demanding the place way of modifying two given figures in a new one in order to be solved constitute the first-order factor F4. Therefore the results indicate that the ways of figure modification have an effect on operative figure understanding. The first-order factors F5 and F6 are regressed on the second-order factor that stands for discursive apprehension, indicating the effect measurement concept exerts on this type of geometric figure apprehension. To be specific, the first-order factor F5 refers to the verbal problems which demand increased perceptual ability of geometrical figure relations and basic geometrical reasoning, while the first-order factor F6 consists of the verbal perimeter and area problems.

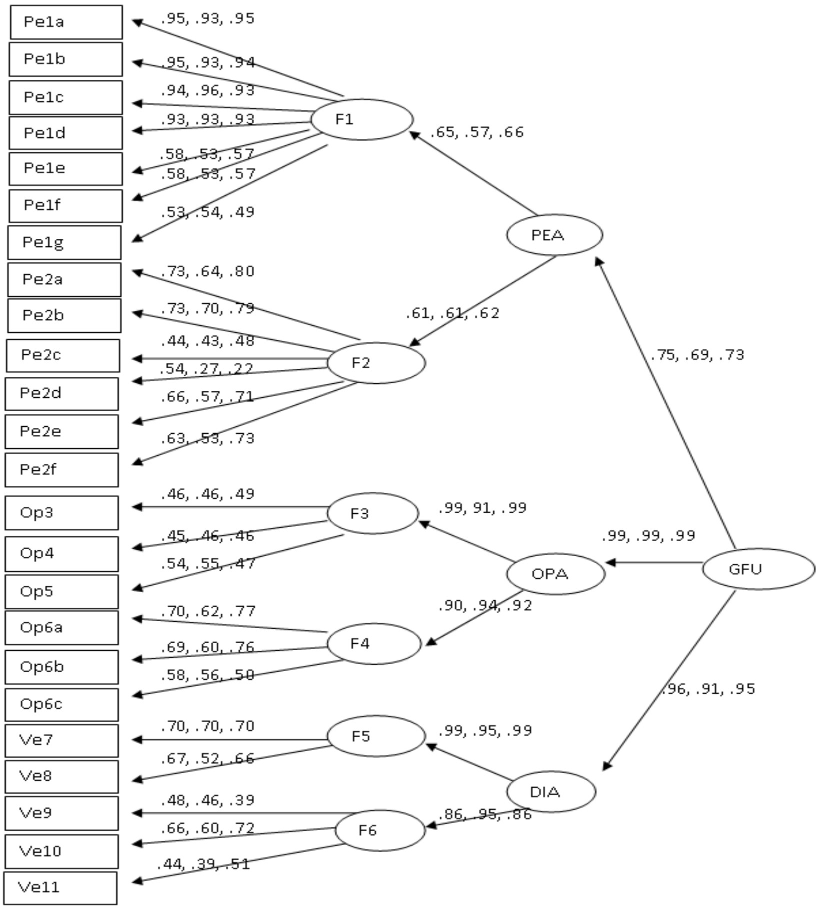

Figure 1. The CFA model of the geometrical figure understanding.

To test for possible similarities between the two educational level groups' geometrical figure understanding, multiple group analysis is applied, where the proposed three-order factor model is validated for elementary and secondary school students separately. The model is tested under the assumption that the relations of the observed variables to the first-order factors, of the six first-order factors to the three second-order factors and of the three second-order factors to the third-order factor will be equal across the two educational levels. The fit indices of the model tested are acceptable [x^2 (485) = 903.78, CFI= 0.97, RMSEA= 0.04, 90% confidence interval for RMSEA= 0.036, 0.044]. Thus, the results are in line with our second hypothesis that the same geometrical figure understanding structure holds for both the elementary and the secondary school students. It is noteworthy that some factor loadings are higher in the group of the secondary school students suggesting that the specific structural organization potency increases across the ages.

The effect of students' educational level. In order to determine whether there are significant differences between primary and secondary school students concerning their performance in the different aspects of geometrical figure understanding, a multivariate analysis of variance (MANOVA) is performed. Table 1 presents the means and the standard deviations for these dimensions in the two educational levels.

Overall, the effect of students' educational level (primary or secondary) is significant (Pillai's $F_{(6,1079)}=34.43$, $p<0.001$). In particular, the mean value of secondary school students' geometrical figure perceptual ability (F1) is statistically significant higher ($F_{(1,1079)}=79.51$, $p<0.001$) than the mean value of primary school students. Similarly, the mean value of secondary school students' recognition ability (F2) is statistically significant higher ($F_{(1,1079)}=38.81$, $p<0.001$) than the mean value of primary school students.

In tasks demanding reconfiguration (F3) secondary school students' performance is statistically significant higher ($F_{(1,1079)}=74.34$, $p<0.001$) than primary school students' performance. In the same way, the mean value of secondary school students' performance in place way modification tasks (F4) is statistically significant higher in comparison with primary school students' performance ($F_{(1,1079)}=36.03$, $p<0.001$).

Concerning primary and secondary school students' performance in verbal problems the results are quite different in the two dimensions. Particularly, in verbal problems 7 and 8 (F5) the performance of secondary school students is statistically significant higher ($F_{(1,1079)}$ =105.38, $p<0.001$) than the performance of primary school students. In contrast, although the performance of secondary school students in verbal problems 9, 10 and 11 (F6) is also higher than the performance of primary school students this difference is not statistically significant ($F_{(1,1079)}=0.03$, $p=0.85$).

Therefore, the above findings verify the third hypothesis stating that differences exist in the geometrical figure understanding performance of primary and secondary school students. In particular, secondary school students' performance is higher in all the dimensions of the geometrical figure understanding relative to the primary school students' performance.

Level	F1		F2		F3		F4		F5		F6	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
Primary	0.45	0.41	0.62	0.26	0.32	0.31	0.31	0.38	0.38	0.40	0.247	0.28
Secondary	0.66	0.38	0.72	0.27	0.49	0.35	0.45	0.42	0.63	0.40	0.251	0.31

Table 1: Means and standard deviations in geometrical figure apprehension dimensions in primary and secondary school students

Similarity diagrams. Figure 2 and 3 present the similarity diagrams of primary and secondary school students' responses to the tasks of the test. Particularly, in Figure 2 two clusters (i.e., groups of variables) can be distinctively identified. The first cluster consists of the variables corresponding to the perceptual tasks (Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g). In the second cluster the variables representing the recognition, operative and verbal problem solving tasks are included (Pe2a, Pe2c, Pe2b, Pe2e, Ve11, Pe2d, Pe2f, Op6c, Ve7, Ve8, Ve9, Ve10, Op6a, Op6b, Op3, Op5, Op4).

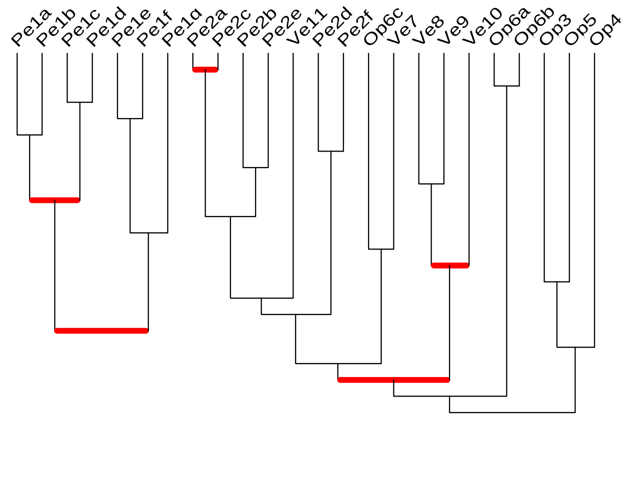

Figure 2. Similarity diagram of primary school students' responses to the test

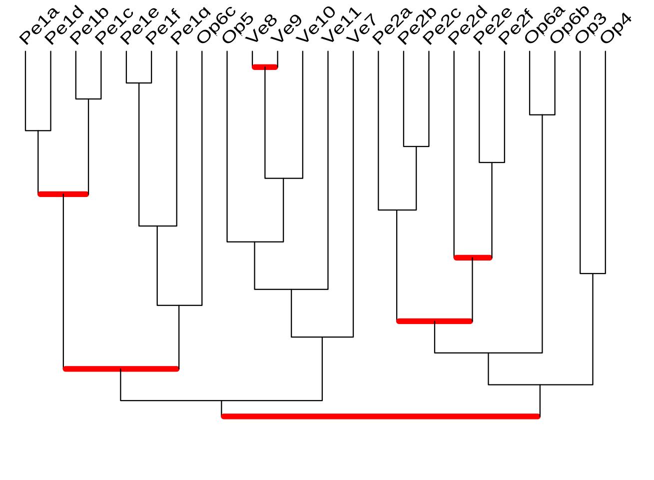

Figure 3. Similarity diagram of secondary school students' responses to the test

In Figure 3, three clusters can be identified. The first cluster includes the perceptual tasks and an operative task (Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g, Op6c). The second cluster consists of an operative task and the verbal problem solving tasks (Op5, Ve8, Ve9, Ve10, Ve11, Ve7). The third cluster involves the recognition tasks and some of the operative tasks (Pe2a, Pe2b, Pe2c, Pe2d, Pe2e, Pe2f, Op6a, Op6b, Op3, Op4). Comparing the two diagrams some relations between the variables remain invariant indicating a stability of the way the primary and secondary school students behave during their solution process (e.g. Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g and Ve8, Ve9, Ve10).

However, differences are observed in many relations of variables. For instance, primary school students behave in a similar way during the solution of the recognition and verbal problem solving tasks, while secondary school students behave in a similar way during the perceptual, some operative and verbal problem solving tasks. Furthermore, in Figure 3 the three clusters are strongly connected with each other indicating that secondary school students behave in a consistent way during the solution of the perceptual, operative and discursive tasks. In contrast, primary school students deal with perceptual tasks in isolation indicating a compartmentalized way of thinking (Duval, 2002). The similarity diagrams' results provide evidence for differences in the way primary and secondary school students behave during the solution of the perceptual, discursive and operative apprehension tasks, verifying the fourth hypothesis.

CONCLUSIONS

This study investigated the role of perceptual, operative and discursive apprehension in geometrical figure understanding. Structural equations modelling affirmed the existence of six first-order factors indicating the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concept, three second-order factors representing perceptual, operative and discursive apprehension and a third-order factor that corresponded to the geometrical figure understanding. It also suggested the invariance of this structure across elementary and secondary school students. Thus, emphasis should be given in all the aspects of geometrical figure apprehension in both educational levels concerning teaching and learning.

Furthermore, differences existed in the geometrical figure understanding performance of primary and secondary school students. Particularly, secondary school students' performance was higher in all the dimensions of the geometrical figure understanding relative to the primary school students' performance. The performance improvement can be attributed to the general cognitive development and learning take place during secondary school. In fact, secondary school curriculum in Cyprus involves many concepts already known and mastered during primary school. This repetition of concepts leads to higher performance even though primary and secondary school instructional practices differ.

Concerning the way students behaved during geometrical tasks solution process it was observed that the behaviour of primary and secondary school students was similar during the solution process of some of the tasks. This finding revealed that geometrical figure understanding stability existed to a certain extent in these students' behaviour. However, in some cases differences were observed in the way the two age groups of students dealt with geometrical figure understanding tasks. To be specific, secondary school students behaved in a consistent way during the solution of the perceptual, operative and discursive tasks. In contrast, primary school students dealt with perceptual tasks in isolation indicating a compartmentalized way of thinking. In fact, the results provided evidence for the existence of three forms of elementary geometry, proposed by Houdement and Kuzniak (2003). We may assume that in this research study, primary school teaching is mainly focused on Geometry I (Natural Geometry) that is closely linked to the perception, is enriched by the experiment and privileges self-evidence and construction. On the other hand, secondary school teaching gives emphasis to Geometry II (Natural Axiomatic Geometry) that it is closely linked to the figures and privileges the knowledge of properties and demonstration. As a result, in the case of primary school students geometrical figure is an object of study and of validation, while in the case of secondary school students geometrical figure supports reasoning and "figural concept" (Fischbein, 1993).

It seems that there is a need for further investigation into the subject with the inclusion of a more extended qualitative and quantitative analysis. In the future an investigation of the way students who master perceptual, operative and discursive apprehension behave in complex activities that require a coordinated approach to these geometrical figure understanding dimensions should be conducted. It would be also interesting to compare the strategies primary and secondary school students use in order to solve perceptual, operative and discursive apprehension tasks. Besides, longitudinal performance investigation in geometrical figure understanding tasks for specific groups of students (e.g. low achievers) as they move from elementary to secondary education should be carried out.

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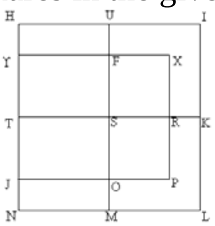
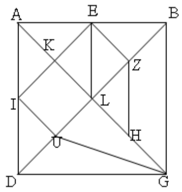
APPENDIX

1. Name the squares in the given figure:



(Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g)

2. Recognize the figures in the parenthesis (KEZL, IEZU, EZHL, IKGU, LGU, BIL)



(Pe2a, Pe2b, Pe2c, Pe2d, Pe2e, Pe2f)

3. Underline the right sentence:

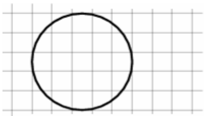


Figure 1

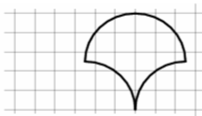
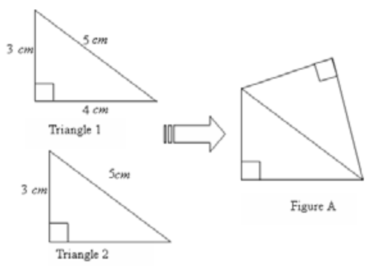


Figure 2

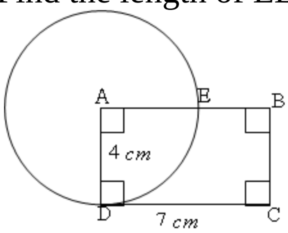
(Op4)

a) Fig. 1 has equal perimeter with Fig. 2
b) Fig. 1 has smaller perimeter than Fig. 2
c) Fig. 1 has bigger perimeter than Fig. 2

4.Peter combines Triangle 1 and Triangle 2 making Figure A. Calculate the perimeter of Figure A. (Op6a)



5. In the following figure the rectangle ABCD and the circle with centre A are given. Find the length of EB.



(Ve7)

6. Themistoklis has a square field with side 40m. He wants to construct a square swimming pool which is far from each side of the field 15m. Find the swimming pool perimeter. (Ve9)

GESTALT CONFIGURATIONS IN GEOMETRY LEARNING

Claudia Acuña

Cinvestav-IPN, Mexico

ABSTRACT

The treatment of geometric diagrams requires the handling of the figural aspects of the drawing as much as the conceptual aspects contained in the figure[1]. In geometry we use the figural aspects of diagrams as symbols to prove or resolve problems. When we interpret figural information, what we call Gestalt configurations emerge: auxiliary figural configurations, real or virtual, that give meaning and substance to an idea that facilitates the proof or solution to the problem. In this work we give arguments to acknowledge the existence of these resources, identify their symbolic nature and consider the reasons behind their existence, sometimes ingrained, sometimes superficial.

INTRODUCTION

To conceive representation as "one thing in place of another, for someone" Pierce (1903) allows us to interpret it as a semiotic mediator between the abstract object of study and the cognizant individual.

In this sense the symbolic aspect in terms of the syntax of the representation must be considered as much as its semantics. The semantics are grasped by the individual through meaningful problematic practices.

In this work our aim is to identify the role played by the auxiliary constructions related to the use of diagrams, which we call Gestalt constructions and which are built by the users when they figural manipulate drawings in order to treat them as figures, Laborde and Caponni[2], (1994).

We hold that these configurations are profoundly ingrained in our students, that they are intentional but often unstable. They can be a particularly valuable resource in heuristic tasks of figural investigation.

THEORIC FRAMEWORK

From the point of view of Duval (1995):

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[1] In the sense of Laborde and Caponni

[2] The treatment of the graph as a drawing or figure, is based, firstly, on observing its properties as an actual pictorial representation or, secondly, considering the mathematical properties associated with the graphical representation.

> **One figure[3] is an organization in marked contrast to the shine. It emerges from the background through the presence of lines or points, governed by Gestalt law and perceptual indications** p.142

In terms of the Gestalt relationship the figure has "form, contour, and organization," while its preceding appears as an "amorphous and infinite continuity", Guillaume (1979) p. 67.

Pictorial representations may be considered external and iconic, Mesquita (1998); they are also defined as inscriptions, Roth & McGinn (1998); or diagrams, Pyke (2003). The unifying idea is that the graph is an external representation that is materialized through the use of pencil and paper, the computer or other means and is, therefore, available through these means, in contrast to mental representations which are not accessible, op cit.

Below we consider the graphic representation as a diagrammatic representation or diagram that preserves the relationships of the objects involved. Diagrams from the viewpoint of sense will be observed in themselves and interpreted from the point of view of the reference between them.

On the other hand, diagrams are figural concepts that, in the words of Fischbein (1993) can be thought of as concepts and as objects: this duality emphasizes the different interpretations associated with graphic representations.

Thinking of a diagram as an object means associating specific figural properties with it, such as position or form. These considerations on what thinking about it as an object means, in Fischbein's way, refer to a mathematical object, this is abstract. The dichotomy between object and concept is related more to a theory need to include non-formalized mathematical aspects, such as position or form, than to the mathematical objects in themselves.

For the purposes of this work we refer to the treatment of representations in geometry based on their iconic or figural properties centered on visual image and to their external nature as embodied materially on paper or other support.

The nature of diagrams in geometry learning is ruled by two types of properties as Laborde (2005), observes:

> **Diagrams in two-dimensional geometry play an ambiguous role: on one hand they refer to theoretical geometrical properties, while on the other, they offer spatial-graphical properties that can give rise to a student's perceptual activity** p. 159

The treatment given to the diagram as an object in geometry learning is closer to that given to a drawing as a current instance, and not as an

[3] The word "figure" in this quote has a meaning close to diagram, distinct from how we use it in the rest of the work.

abstract mathematical object in the concept-object duality. It takes students some time, in fact, to incorporate the idea that drawn objects (representations) have properties which are distinct from those of real life objects.

In terms of learning, Laborde op cit. warns:

> **The distinction of the two domains, the spatial-graphical domain and the geometrical one, allowed us to show that the intertwining of the spatial aspects of diagrams with the theoretical aspects of geometry is especially important at the beginning of learning geometry** op. cit. p. 177.

It is in the spatial-graphical domain where spatial and figural relations are developed that give shape to the thought structures that are developed around the Gestalt. First, as a relation between the background and the form and later, as resources in the explanation, construction or solution of problems, they give rise to Gestalt configurations.

Studies related to visualization and, most recently, visual perception, have addressed the role played by Gestalt relations between background and form in the pictorial representation that accompanies the mathematics, and the importance of considering it on a certain type of perceptive perception, Duval (1995)

In the work of Nemirovsky and Tierney (2001), regarding spaces of representation, we observe a special interest in establishing the existence of distinct ways of interpreting the same space of representation based on its use and meaning relative to the objects represented.

From the above we can say that the use of diagrams depends not only on what is represented in them, but also on the relations we can establish from them, including spatial information which includes Gestalt relations.

Gestalt configurations

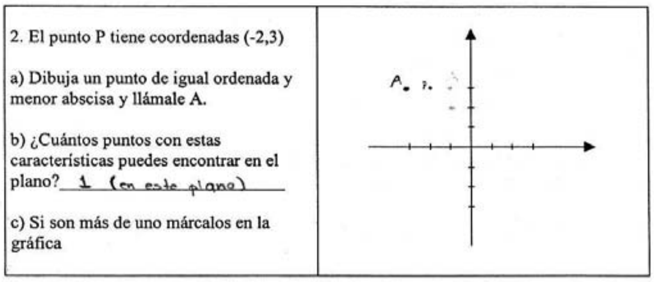
In the work of Dvora and Dreyfus (2004) we have unjustified assumptions based on diagrams in geometry due to students confusing a mathematical motive and a purely visual motive. In addition, when problem solving they base themselves more on their beliefs about the topic in question than on the available propositions. The authors find that diagrams affect students' way of thinking because, among other things, they use diagrams as evidence.

The Gestalt configurations dealt with here have no evidential connotation, they are, instead, auxiliary constructions that complete or give shape to an idea and have their origin in the need to solve problems which involve a diagram.

Gestalt configurations are not related to all the possible pictorial tests that claim to find a solution helped by the drawing, whether the lead is promising or not.

A Gestalt-type configuration, as well as the intentionality of solution, should contain a reference to the relation between background and form, that is, Gestalt configuration "adjusts" to the general composition of the diagram. In other words, Gestalt configuration manifests as a cognitive resource to give substance to a thought and is distinguished by its figural relation between the background and the form of the diagram in question.

The symbolic relations of a Gestalt configuration are determinant: it is dependent on them whether this configuration can be built or not. By way of example, Acuña (2004), we have the case in which without the presence of a graphic reference the very existence of the geometric or graphic object is in doubt, as in the following cases:


2. El punto P tiene coordenadas (-2,3)

a) Dibuja un punto de igual ordenada y menor abscisa y llámale A.

b) ¿Cuántos puntos con estas características puedes encontrar en el plano? 1 (en este plano)

c) Si son más de uno márcalos en la gráfica

Fig. 1 Point A is the only one with equal ordinate and smaller abscissa than P, in this plane

In the student's answer to the question about the number of points that have an equal ordinate and smaller abscissa than the point (-2,3) in which he (or she) affirms: 1 on this plane*,* we can see that he is trapped by the actual representation since the picture offers only one unit mark on the abscissa axis. The student does not consider alternative solutions other than that point located above the mark of the whole abscissa unit. The absence of the mark combines with the idea that a point should have a whole abscissa unit. This student was unable to build neither of a suitable configuration for the solution or a Gestalt configuration.

In the following case, Acuña (1997) we have (see Figure 2) a question about whether the suggested points are on the drawn straight lines or not. If we look at the point (-2, 3) we see that the straight line proposed does not reach the position where a perceptive solution could be given, that is, one perceived "by eye". This fact makes the student doubtful and answers that if we lengthen the straight lines, the point is on it, otherwise it isn't.

Our student is unsure of the existence of the point in spite of knowing its coordinates, thus the Gestalt configuration cannot be built because of the absence of the graphic reference that gives it substance. In this case, if the straight line does not reach the indicated place, there is no security about its existence, which impedes the acceptance of the relation between the straight line and the point.

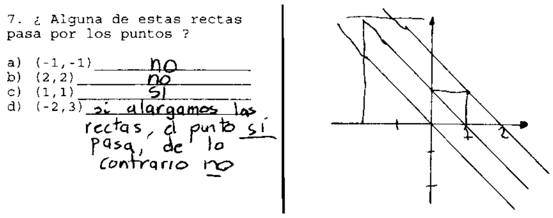


Fig. 2 Problem of points on the straight lines

Constructions with appropriate Gestalt configuration

In relation to the construction and use of geometric figures, Maracci (2001) has observed that students insist on making constructions that possess certain, from their point of view, appropriate aspect.

This insistence is accompanied by the preference for the horizontal-vertical position, or the choice of graphs that appear to be, for example, a straight line Mavarech and Kramarsky, (1997) or a segment of a straight line with an slope equal to 1, Acuña (2001), as well as students' penchant for using prototypes[4] Hershkowitz (1989), or the use of the "best" examples from among one same category of possible cases, Mesquita (1998).

This phenomenon can be explained by the students' need to find a good orientation and familiar representation. In other words, they prefer to build "appropriate" configurations in general and Gestalt configurations in particular that give meaning to the actual figural relation.

In some tasks with qualitative instructions, as in figure 3, we have identified a tendency to recognize and build graphs in a certain position and with a certain peculiarity, forming prototypes, Acuña (2001). A large part of the students surveyed with the question for draw straight line with only points with positive abscissa, responded with a half-line that reaches the origin, with a slope of 1. This answer was more frequent than any other, correct or incorrect, in high school students.

[4] We call prototypical figures those which correspond to a regular organization of contour, orientation and form; prototype figures tend to respect laws of enclosure (closed limits are preferably perceived), favoring some directions (such as horizontal and vertical) and forms (that tend to be regular, simple, and symmetrical); the components of the figure (sides, angles for example) have approximate dimensions.

5. Draw a straight line where all the points have a positive abscissa, that is, where $x > 0$ is true for all points on the line.

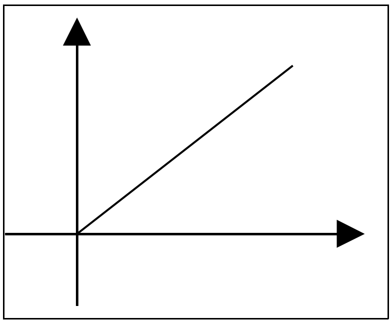


Fig. 3 Answer to a qualitative-type construction task

The students' answer presupposes that the straight line built does not cross to the other side of the vertical axis, as if it were a barrier, so that it will not take negative values for the abscissa.

The non-ostensive nature of the straight line related to the infinite extension of its extremes contributes to the incorrect interpretation of the answer that, in strictly figural terms, has a plausible logic, especially since it is not possible have a representation of a straight line, only parts of it.

The non-ostensive aspect on the infinite extension of the line can be accepted theoretically by the students, but the impossibility of building theoretical straight lines leads them to accept the segments of a straight line as if they were straight lines themselves.

In figure 4, Acuña, (2002) students are asked to draw the graph of the straight line that would have an ordinate equal to the origin of the original straight line that appears on the left.

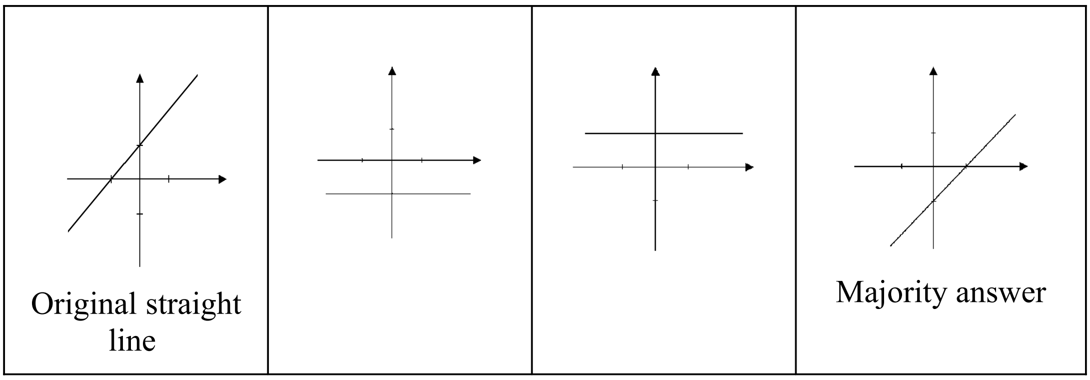


Fig. 4 Gestalt configuration combining figure and form

The majority of our students drew the graph on the far right. Many of them had correctly recognized the ordinate of the origin in straight lines given earlier; nevertheless, here they choose to conserve the "triangular" image formed in both graphs, preferring to relate the two graphs with a similar Gestalt.

This type of answer is strongly conditioned by the situation of the exercise, in particular given that this perception is unstable, as we can see in other exercises.

In the following exercise, Sosa (2008) two high school students have been asked to build the height corresponding to the side marked with X in each case.

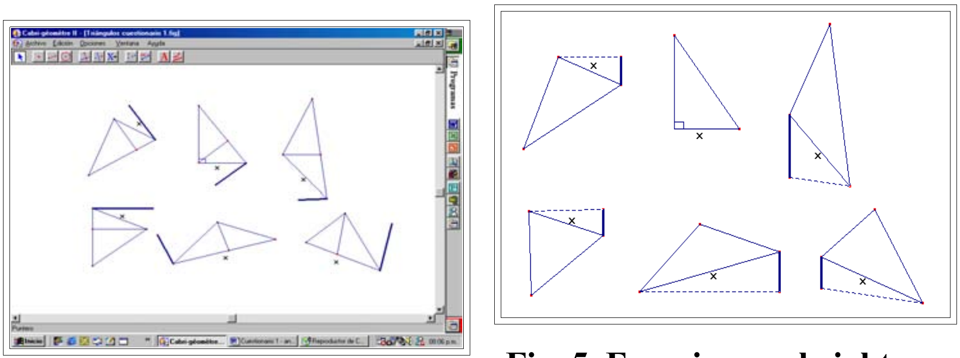


Fig. 5 Exercises on height construction

In these two cases, we have the application of a Gestalt configuration to solve the problem of the construction of the height of the marked side. In the answer on the left, the height is thought of as a conformation formed by the vertex of the obtuse angle, or what looks like it. The student also uses an auxiliary parallel line which we suppose was in the image the student recalled.

In the case of the constructions on the right (see figure 3) we have an auxiliary construction that includes the line marked with X but where this is a part of another auxiliary construction that presents a right-angle triangle where we observe some of the characteristics relevant to height, but its construction is unknown. The marked line is included in his construction, but its role in the construction is reinterpreted and he does everything he can to make it look good.

In the following case we ask students to mark the straight lines with a different slope to that of the one given.

The formation of this configuration not only appears when the definitions of the geometric objects are unknown or is recalled inexactly, but also when globalizing an idea of position, as in the following example. In the case of figure 6 and 7, we ask high school students to choose from the lower graphs that which have a different slope to the one proposed initially.

The results allow us to see their idea of a slope in this exercise. Despite having correctly compared, based on perception, the slope of the given lines, here they conceive it as the Gestalt configuration formed by the position of the straight line relative to the axes, that is, the line is positioned from left to right and from up to down.

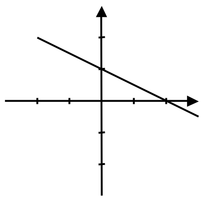


Fig. 6 Straight line with given slope

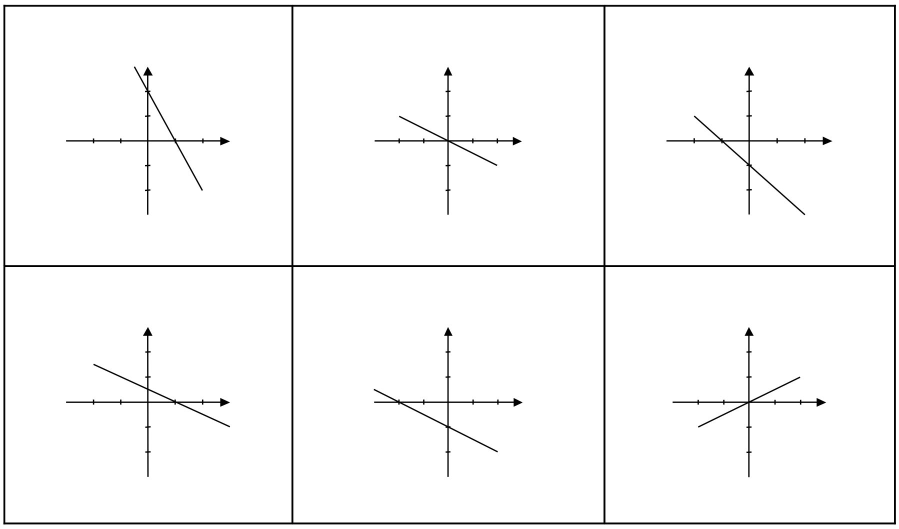


Fig. 7 Gestalt configuration on a slope

The 19.3 % of our sample only marked the straight line that is positioned from left to right, leaving aside the idea of slope that they used before.

The preference towards a "good" Gestalt appears to impose itself in tasks of identification of figural properties. This recourse may signify an advance or a backward step for solution strategies. What does appear to be constant is the use of this type of configuration to test solutions to problems with diagrams.

These configurations may disappear quickly with better instruction, but they also have aspects of profound rooted as in the case of Moschkovich's (1999) investigation, regarding the use of the y-intercept. She finds that when observing the graph of a straight line students may expect the x-intercept to appear in the equation because on the graph it is a salient as y-intercept although this is not necessarily convenient in the case of the equation $y = m\ x + b$ however, they are important for the equation that considers two points on the straight line. The appeal of the x-intercept is so big than could think it as a preconception; in her investigation she affirms that:

> **The use of x-intercept is not merely the result of choosing or emphasizing the form y = m x + b over other forms but is instead an instance of students making sense of the connection between the two representations and reflection on the conceptual complexity of this domain** p. 182

We believe from the above that it is possible to suppose the existence of figural resources that take the form of Gestalt configurations that respond on one hand, to the necessity of giving substance to figural ideas, and on the other, that these configurations are ruled by the relations between background and form on which rests the figural representation of mathematical and, more concretely, geometric diagrams.

CONCLUSIONS

A Gestalt configuration is a mental or real construction utilized by the user to resolve, complete or give meaning to a given problem through a diagram that can be treated as a drawing or figure.

Gestalt configurations have a personal character, but on occasions reflect epistemological obstacles that are supported by the non-ostensive nature of the properties of the objects represented by the diagrams, as in the case of the infinite character of some of these representations.

The formation of some Gestalt configurations is characterized by having an ephemeral life, although there are some that persist; as they are personal productions of the user. In general, they are considered productive and reliable for confronting familiar graphic settings towards resolving problems that include diagrams.

In all cases, the construction of the Gestalt configurations is intentional in spite of the inability to ensure its pertinence. Gestalt configurations do not only appear as visual traps but as a diversity of resources to solve figural problems or proving.

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INVESTIGATING COMPARISON BETWEEN SURFACES

Paola Vighi

Mathematics Department – University of Parma – Italy

This work is based on a geometrical problem concerning comparison between surfaces, presented to 58 pupils 10-11 years old. We present a worksheet aimed at revealing children's reasoning about visualisation in geometry. We compare the ways in which various problems are tackled by two different groups of students: Group E (experimental) and Group T (traditional). We conclude with some observations about teaching geometry and suggestions for its improvement.

INTRODUCTION

During a lecture to future teachers about fractions, I observed as they were analysing suitable geometric figures, drawn using computer graphics. I realised that these drawings could be useful for investigating geometrical learning. My attention was particularly attracted by different representations of the half of a rectangle. I mentioned my idea to a group of experienced Primary School teachers, and one of them, when she saw figures A, B and C (Figure 1), said: "If the pupils have already worked with fractions, they will certainly use and recognize the concept of half." As in my experience this conclusion is rash and not entirely obvious, I decided to investigate it. Working with the teachers, we prepared a worksheet based on Figures A, B, and C and on a fourth Figure D, expressly created.

The aim of the research is twofold: to investigate the use of the concept of 'half,' and chiefly to study geometrical thinking observing pupils behaviours, with particular reference to registers of representation (Duval, 1998-2006), especially the figural register.

THEORETICAL FRAMEWORK

The concept of *half* and related notations are present in five and six-year-old children (Brizuela, 2006). At this age, children use different semiotic representations (Duval, 1995), but it is difficult for them recognise a half in different representations (Sbaragli, 2008). According to Duval, the passage from a semiotic representation to a different representation is fundamental for a conceptual learning of objects. In particular, he distinguishes two possible kinds of transformation of representation: *conversion* (from a semiotic representation to another, in a different register) and *treatment* (from one semiotic representation to another, in the same register). The half of a geometrical figure is usually presented to children when we introduce fractions, as one of the first examples. Subsequently, teachers move on to writing fractions and to calculating with them, moving from conversions to treatments.

Traditionally in Primary School we use geometrical figures as a suitable tool for teaching and learning geometry. Figures involve a fundamental action for the pupil: looking. The didactical contract (Brousseau, 1986) based on showing requires that

> "the pupil understands what the teacher expects that s/he will see, with the false illusion that both must see the same" (Chamorro, 2006).

If both parties do not see the same, the contract is broken and learning does not take place. So we need to … "teach to see". In geometry, a first problem is created by *perception***,** which may hinder the ways of seeing figures. In other words, the perceptive indicators may be misleading for the qualitative evaluation of the extension of surface or of other magnitudes. Gestalt theory deals with laws of organisation of visual data that lead us to see certain figures rather than others in a picture.

More recent researches show that

> "…it is the task that determines the relation with figures. The way of seeing a figure depends on the activity in which it is involved." (Duval, 2006).

Duval (2006) analyses and classifies the different ways of seeing a figure depending on the geometrical activities presented to pupils. He distinguishes four ways of visualising a figure: by a botanist, a surveyor, a builder or an inventor. Botanists and surveyors have 'iconic visualisation', and perceive the resemblance between a drawing and the shape of an object. Builders and inventors on the other hand have 'non-iconic visualisation', and their perception is based on the *deconstruction of shapes***.** Duval analyses the introduction of supplementary outlines, which he thinks fundamental in 'non-iconic visualisation', in particular he discusses *re-organising outlin*es which allow to reorganise a figure and thus to reveal in it parts and shapes that are not immediately recognizable. .

He also discusses the *méréological decomposition*[1] of shapes, a division of the whole into parts which can be juxtaposed or superimposed, with the aim of reconstructing another figure, often very different to the starting figure. This allows the detection of geometrical properties needed to solve a problem, using an exploration purely visual of the figure initial. He distinguishes three kinds of méréological decomposition: material (with cutting and rebuilding as in a jigsaw puzzle), graphic (using reorganising outlines) and by looking (with the eyes, not "mentally"). We tackled the problem of "which is 'visual' in geometry?" in a research paper (Marchini *et al.*, 2009) where we analysed in-dept the literature on this argument.

In Italian Primary School, *comparison between surfaces* is often reduced to evaluating areas (measurements of extension of surfaces) and to comparing numbers. Teachers tend to determine equivalence of the magnitude of two objects by means of measurement. But "transferring the comparison to the numerical field, we are in fact working with numerical order which doesn't consider the criterion of quantity of

[1] In mathematical logic, mereology is a theory dealing with parts and their respective whole. The term was coined by Łésniewski in 1927, from the Greek word μέρος (*méros*, "part").

magnitude" (Chamorro, 2001). An epistemological slide from geometry to arithmetic occurs. The comparison between surfaces and, in particular, the "equivalence of magnitude" is a fundamental but difficult concept, which requires specific teaching. In previous research we wrote:

> "We did not predict that determining shapes of the same area would be difficult, …. But in fact there were cases where pupils failed to recognise that two congruent rectangles, set at a different way on the sheet of paper, had the same extension." (Marchetti et al., 2005).

The comparison between surfaces is also influenced by the relationship between *shape and surface*: when we present a surface, we present something that has a specific shape. If the shape changes, a younger child might think that the surface changes too. Research shows clearly that pupils under 12 have difficulty in understanding that the shape and the surface of a figure are different (Bang Vinh & Lunzer E., 1965).

RESEARCH METHODOLOGY

We presented the worksheet at the end of the last year of Primary School, to three classes of students 10-11 years old, which had followed two different approaches to geometry. One class had already taken part in an experimental project and the other two classes had received only traditional teaching. We named the first group 'Experimental' (Group E) and the second group 'Traditional' (Group T). Group E consisted of 26 pupils; they had followed a Mathematics Laboratory Project (MLP)[2], during the last three years of Primary School. It focussed on activities that started from a practical problem, such as fencing in a field or tiling a room, and led to the introduction of specific instruments by the teacher as the children perceived the need for them. The early activities involved concrete materials and children using their hands, and geometric instruments and theoretical concepts were introduced in later activities. So Group E did not follow traditional curricular teaching; we presented new activities that were different in terms of both methodology and content. Group T consisted of 32 students from two classes which had followed the traditional mathematics curriculum. Both groups had previously studied and worked with fractions and areas. For Group E, however, the project had opted to present area before perimeter, which is unusual in Italian schools.

Pupils' behaviours were observed as follows: when the teacher presented the worksheet, s/he explained that not was possible to use a rubber, but if necessary children could write their notes and opinions on another sheet of paper. I then analysed the protocols.

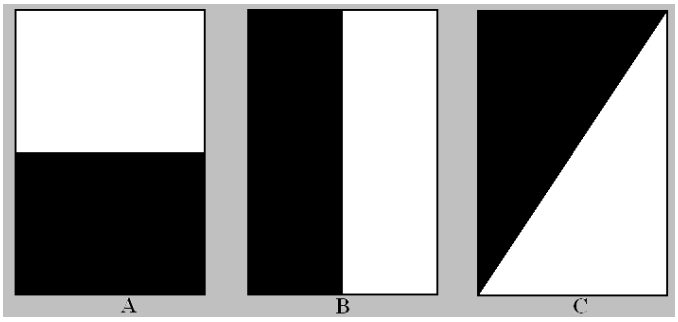
[2] The project was carried out by two researchers, D. Medici and P. Vighi, and two teacher-researchers, P. Marchetti and E. Zaccomer.

THE TASK AND ITS ANALYSIS

In the following pages we present and discuss the worksheet.

A pizza-maker with a lively imagination displays these slices of pizza.

All the slices have one part with only tomato (dark) and one part with only mozzarella (light).



One child wants a slice of pizza with a lot of tomato.

Which slice do you think he or she should choose? Why? ....

Does the slice of pizza below have more mozzarella or more tomato? ....

Why? ....

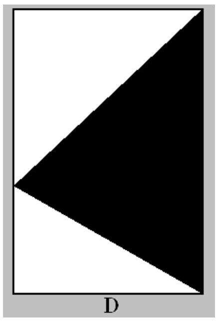


Figure 1: the worksheet

This activity on geometrical figures in the first part lies on the first level of van Hiele's theory, in the final part it lies on the second level, which involves the possibility of seeing inside geometrical figures and seeing and/or making a subdivision into parts (van Hiele, 1986). In the paradigmatic perspective introduced by Houdement and Kusniak (2003), the activity is situated in Geometry I.

Notice that the passage from A to B or C requires 'treatments' inside the register of visual representations. The first question is deliberately ambiguous; the form of the question could lead the child to opt for only one of the slices and, consequently, give a wrong answer. In other words, the question could lead the child to exclude the equivalence of surfaces. The second part of the task presents an unusual geometrical problem. The slice is divided into three parts and the comparison concerns only two quantities of food (two surfaces). There is a different subdivision in half of the same rectangle as before. The question is formulated differently from the first: the problem is the comparison between tomato and mozzarella. Using a supplementary outline helps to find the answer. The main information is in the drawings: rectangles A, B, C and D are congruent (8 cm $\times$ 5.3 cm) and, in particular, in A and B we used the middle point of a side, without specifying this; in other words, we gave implicit data. Figures play an essential role: they are shown against a grey background, with the aim of distinguishing between the whole slice and its parts.

The context of the problem is intended to focus attention on surfaces. The figures in the first part, rectangles and triangles, are familiar; the pupils know the formulas for the calculation of their areas. The last ‘slice’ is made up of a dark triangle, representing tomato, and two other white triangles, not contiguous, representing just mozzarella. It is an unusual figure which does not appear in textbooks (it may not in fact appear in pizza shops either), but if the sheet of paper is rotated, it probably becomes more familiar as a drawing related to the formula of area of a triangle. For Figure D too, children need to use the concept of half, or they need to “see” congruent parts, or draw supplementary outlines, or calculate areas and verify their equality.

The analysis of A and B by *méréological decomposition* is simpler than for C. In effect there is a difference in the geometry of transformations: in A and B it is sufficient to translate some pieces, while in C rotation is also required. As we saw, D implies cutting the figure and reconstructing congruent parts. We present slice D to investigate pupils’ strategies. We want to establish whether children use the same methods for answering both questions, or if D encourages them to try different methods. We also want to observe whether solving the second problem leads pupils to rethink their answers to the first.

RESEARCH RESULTS

The activity is presented in a geometrical context, which often seems to imply the use of specific geometrical tools. In many of the protocols the shift from the geometric register to the numerical register of fractions does not occur: ‘conversion’ between the registers does not take place.

Only a few answers to the first question (12% in Group E, 6% in Group T) use the concept of “half”: “Figures are divided in half”, or “Half the space is filled with tomato”. The question draws pupils’ attention only to the black shapes, or tomato. In other words, children focus on and compare particular parts, rather than looking at the slices globally. It is not by chance that the few answers which are based on “half” make recourse to the relation part-whole (Hart, 1985): “All slices are perfectly divided in the middle and the whole is equal for all figures”. Notice that the children use words that are usual in speaking about fractions, not the symbol 1/2. In some cases the concept of half is questionable and ‘relative’: “I choose pizza C because tomato occupies the “biggest half.” The relation shape-surface also emerges: “Even if the pizzas are divided into different shapes, it is still half a slice and the slices are equal”.

The "equal extension" of tomato surfaces in A, B, C was recognised by only 6 pupils in Group E and 4 in Group T.

We now analyse different procedures observed for the first part of worksheet.

- *by perception*: children choose slice C because the tomato appears bigger (or "It looks like a piece of pizza") (30% in E and 37% in T). In some cases, the choice is based on exclusion, which may be due to the question: some children verify that A and B have equal quantities of tomato, and they conclude that C must be bigger, without checking. Two pupils choose A because "it is larger," taking account of one dimension only.
- *by subdivision*: here we notice very different behaviours according to the teaching methods adopted. In Group T, only 1 pupil uses méréological decomposition, while in Group E 6 do so. Pupils divide figures B and C by drawing (graphic decomposition) or imagining (decomposition by looking) a continuation of the horizontal line present in slice A which divides the white and black parts. They observe that it is possible to shift some black pieces of B or C in order to obtain A. It is significant that some of them write "If I cut in half …", although they did not see the half in Figures A, B and C.
- *by calculation of area*: only 4 pupils in Group E and 3 in Group T calculate 21.20 cm^2 as measure of three surfaces covered by tomato. There is also a problem of approximation: for figure B, in calculating 5.3 : 2 they stop at the first digit after the decimal point obtaining 2.6 and 2.6×8 make 20.8. Slice B thus seems to have less tomato.
- *by calculation of perimeter*: 6 children in Group E use this method (maybe because perimeter was most recently studied) and 5 in Group T. Their procedures are based on measuring the sides of the black figures and their addition: in this way C appears biggest. This is a manifestation of perimeter-area conflict. (Chamorro, 2002), (Marchetti et al. 2005).
- *by flooring with squares*: based on reproduction of figures on squared paper, often without respect for shapes and measurements, or based on the superimposition of a squared grid, often not regular. Answers are based on counting the number of squares.

In the second part of the worksheet, we recorded 58% correct answers in Group E, and 34% in Group T. Obviously the use of half in the first part of the task is a successful strategy, as it is for the second part.

In Group E, previous methodological decisions and their experience of manipulation led children to tackle the problem in different ways. Some children took scissors, cut the pieces and superimposed two white pieces on the black. They still worked with real and not geometrical objects. Their conclusions may be "They are equal," or not, because there is a problem of approximation: "They differ by a small amount". Recourse to *méréological decomposition* promotes fast and correct answers, based simply on the drawing of a horizontal segment, and the height of the dark triangle. An interesting observation is that a few pupils use the expressions “triangle” or “height of triangle” in their explanations; they write: “I connected the vertex of triangle with the opposite side …” or “I drew a horizontal line …”.

Some pupils make a rough estimate, and make recourse only to perception (26% in Group E, 40 % in Group T). They support their answers as follows: “I can see it,” “The part with tomato is slightly bigger.” In some answers the decision is based on the number of pieces, not on areas: “Mozzarella, because two pieces occupy more space than one.”

Both groups make little use of calculation. One girl wrote: 5.3 × 8 = 42.4 and 42.4 : 2 = 21.2 tomato piece; 5.3 × 5 = 26.5 and 26.5 : 2 = 13.25; 5.3 × 3 = 15.9 and 15.9 : 2 = 7.95; so 13.25 + 7.95 = 21.20 mozzarella piece. This is an example of rigorous application of rules, without geometrical reasoning.

Another boy uses ‘pre-algebraic’ notation and reaches an incorrect conclusion based only on intuition or perception. He tries to explain (Figure 2) that, starting from the area of the rectangle, we can subtract the areas of two white triangles and we obtain the area of the big triangle (black). In the second part, he observes that the sum of the areas of the white triangles is bigger than the area of the ‘big triangle’, but he doesn’t explain why.

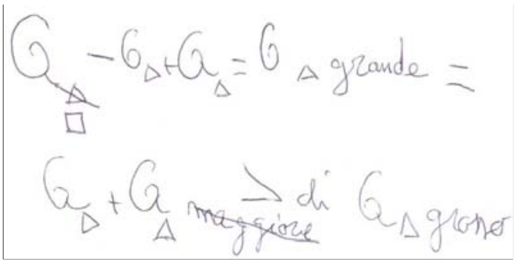


Figure 2: pre-algebraic notation

Some pupils measure two or all sides and multiply them: the idea of multiplication in area calculation is strong, which may be a result of the didactical contract, but there is no understanding of its meaning. We also find mixed procedures: (8 × 5.3) – (8 + 6 + 7) = 42.4–21 = 21.4 area tomato, 42.4–21.4 = 21.0 area mozzarella; the idea is to subtract from the rectangle area the dark triangle area, but the formula for finding the area of a triangle seems not to be known and the pupil calculates the perimeter. Nevertheless one child has a good idea: to obtain the white area as complementary to the black in the rectangle. Only this one boy used this strategy: in fact in school we usually present exercises involving only one shape, and the possibility of calculating an area by subtraction is not introduced.

The solution based on *méréological decomposition* appears the best, and is a successful strategy especially in Group E. We presume that the previous work with Tangram and a different methodological approach helps in the case of Figure D and its parts. Reasoning is based on the use of a supplementary outline (Figure 3).

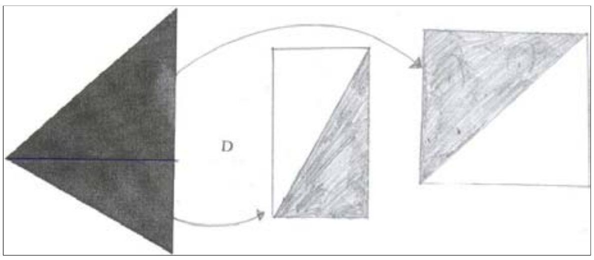


Figure 3: méréological decomposition

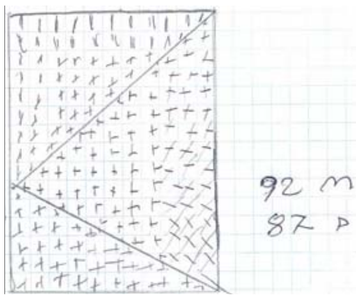


Figure 4: flooring with squares

The idea of measuring with squared paper also appears. In particular, in the protocol reproduced in Figure 4 there is evidence of a lack of understanding: the child counts both squares and pieces of squares and he concludes that the mozzarella area is bigger. In the case of *surface* measurement, schools usually make use of subdivision with squares; there is often no explanation of this method. Moreover it is not suitable for figures with sides that are neither 'horizontal' nor 'vertical'.

Perimeter is used a lot by Group T (18%), but only two pupils use it in Group E (0,07%). It seems that Figure D, which is unusual in traditional teaching, causes the "perimeter-area conflict" and reveals this hidden misconception.

GENERAL CONCLUSIONS

In both groups there were pupils who made no use of geometrical reasoning, but only their eyes. The pizza problem is in fact unusual in that it requires observation of more than one shape and no explicit calculation of its perimeter or area. Often in real life we compare two quantities and we choose the bigger, using common sense rather than mathematics. So one child wrote: "From shapes A, B, and C, I choose C, since it looks like a slice. He was maybe thinking of the shape of a slice of cake. One significant answer came from a child imagining a real pizza, who observed that comparison is impossible, because there is no information about the thickness of the tomato and mozzarella. The analysis of answers confirmed the gap between 'scholastic' and 'real' problems (Zan, 1998). In other words, the same problem presented in the school or a snack bar may have different solutions. Canapés, in fact, are triangular, obtained by cutting a square along the diagonal, and it could well be that we think we are eating more than if the square of bread were cut in other way.

One week later, the teacher of Group E re-presented the worksheet to her class and encouraged a discussion of pupils' own solutions. Many quickly recognized the concept of half as a key to the problem and modified their answers. But some children wrote an explanation clearly without conviction. As we wrote previously, in our experience the concept of half does not seem to have been acquired by pupils 10-11 years old. In our opinion, the concept of half needs to be constructed gradually and it is important to work on it with regularity so that it can successfully prepare the ground for introducing fractions.

We also notice that children often use whole numbers as measures of triangle sides: unfortunately in Italy the problem of approximation is neglected. In some cases pupils understand that different numerical results, can be given simply by approximated measurements, but in other cases the children are closely tied to numerical results, even where this conflicts with common sense.

The global analysis of protocols reveals the influence of different teaching methods. Comparison between the protocols of two groups shows clearly the existence of two different behaviours, closely connected to the "social norm" established in classroom (Yackel, Cobb, 1996) according to the "didactical contract". In Group T, the necessity of following the rules leads to measurement by ruler and the calculation of perimeters and areas. But in Group E, familiarity with manipulation, scissors and so on encourages the use of hands (and the head) (Chamorro, 2008). We observe the presence of an explicit, real geometrical aptitude in Group E, which was probably a result of the MLP. In Group T, traditional geometry and its formulas are prevalent. We surmise that the better results in Group E are closely connected with didactic choices. In other words, the fact that Group E children worked as 'builders' and 'inventors' supports the use of a 'supplementary outline,' which for Duval is fundamental in seeing figures; our experiment confirms his theory of different kinds of visualisation in geometry. Future research will feature an activity based on the same figures but focussing on 'dimensional deconstruction,' defined by Duval as a 'cognitive revolution' for visualisation.

Another important suggestion arises from pupil's approach to the task. Protocol analysis shows that children who use the half or decomposition in shapes A, B and C, use the same concept to investigate D, with the same tools. Vice versa, those who 'found' the half in D, maybe by calculating the area, do not go back to modify their answer to the first part of the task. This points to another critical aspect of traditional teaching, not only in the field of mathematics: exercise books always have be tidy, with no rough work or scribbling, and children are not encouraged to rethink or reflect on work or activities carried out previously. But often sketches and rough drafts can in fact help develop reasoning. We also feel that there should be more encouragement to write up reasoning in the classroom.

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THE EFFECTS OF THE CONCEPT OF SYMMETRY ON LEARNING GEOMETRY AT FRENCH SECONDARY SCHOOL

Caroline Bulf

University of Paris Diderot, Laboratoire André Revuz, France

This paper relates a part of a bigger research from my Phd (Bulf, 2008) about the symmetry's effects on conceptualization of new mathematical concept. We focus here on the results from students' productions at two different levels at French secondary school, with students who are 12-13 years old and 14-15 y.o. We find out different figural treatments according to the transformation at stake. The results work out the concept of symmetry makes students confused with the transformations of the plan at the beginning of secondary school whereas students seem more familiar with metrical properties relative to the symmetry and develop mathematical reasoning at the end of secondary school.

Key word: *secondary school, geometry, transformations of the plan, symmetry, Geometrical Working Space, conceptualization.*

INTRODUCTION

The constructivist wave suggests that a new knowledge is built from the old one. According to the French curricula (1), the symmetry (reflection through a line) is taught since primary school (through folding and paving), and more deeply during the first year of the secondary school (students are 11-12 years old). Next, the rotational symmetry (reflection through a point) is taught during the second year of the secondary school; the translation is taught during the third year and finally rotation is taught during the last year of the secondary school (students are 14-15 y.o.). One of the specificity of the French curricula is to teach the symmetry as a transformation of the plan even if the term "transformation" is mentioned only at the end of secondary school. Others countries (Italy as for instance) deal with transformations of the plan in the frame of the analytic geometry at high school (students are older than 15 y.o). Then, in this French context, we suppose the concept of symmetry takes part into the learning of the new transformations of the plan. The question is **what are the effects of the symmetry on this learning process?** This paper is the rest of our research, already introduced in CERME 5 (Bulf, 2007).

We do not need to argue that symmetry is part of our "real world" but it is a scientific concept too. Bachelard (1934) points out that "nothing is done, all is building", he adds the notion of obstacles "to set down the problem of scientific knowledge". He describes different kind of obstacles: the obstacle of "the excessive use of familiar images", or the obstacle of "common meaning" and "social representations". Nevertheless, we can not ignore the "real world" may be a help for empirical reasoning. As far as our work is concerned, **we wonder if the concept of symmetry**

may be an "obstacle" or a "help" into the learning process of the new transformations of the plan at secondary school. Several French authors have already pointed out some resistant misunderstandings linked with the concept of symmetry (Grenier & Laborde, 1988) (Grenier, 1990) (Lima, 2006) or linked with the others transformations of the plan, and in particular deal with the dialectic global/punctual (Bkouche, 1992) (Jahn, 1998).

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Our research focuses on the process of conceptualization during the learning of the transformations of the plan. The Vergnaud's theory (Vergnaud, 1991), "the conceptual field theory", analyses the human component of a concept in action. We refer to this framework in order to analyse the students who solve mathematical problem. We focus on the adaptation of the "operational invariants" which are actually defined by the concept-in-action ("relevant or irrelevant notion naturally involved in the mathematics at stake") and theorem-in-action ("proposition assumed right or wrong, used instinctively in the mathematics at stake"). The set of these invariants makes the schemes (notion inspired by Piaget) operate. A scheme is the "invariant organization of behaviour for a class of given situation. The scheme is acting as a whole: it is a functional and dynamical whole, a kind of module finalized by the subject's intention and organized by the way used to reach his goal". The "signifiers" *s* (according to Pressmeg's translation of Saussure's meaning (Presmeg, 2006) is the set of representations of the concept, its properties, and its ways of treatment (language, signs, diagrams, etc.). According to Vergnaud, learning is defined as the adaptation of the schemes from students in a situation of reference.

In order to complete the analysis of students' activities through geometrical problems, we refer to the Houdement and Kuzniak's theoretical framework of Paradigm of Geometry I and Geometry II, and the notion of *Geometrical Working Space* (Houdement & Kuzniak, 2006). Geometry I (GI) is the naive and natural geometry and its validity is the real and sensible world. The deduction operates mainly on material objects through perception and experimentation. Geometry II (GII) is the natural and axiomatic geometry, and its validity operates on an axiomatic system (Euclid). This geometry is modelling reality. The notion of *Geometrical Working Space* (*GWS)* is the study of the environment, organized on a suitable way to articulate these three components: the real and local space, the artefacts (as for instance geometrical tools), and the theoretical references (organized on a model). This *GWS* is used by people who organise it into different aims: the *reference GWS* is seen as the institutional *GWS* from the community of mathematicians, the *idoine GWS* is the efficient one in order to reach a definite goal and the *personal GWS* is the one built with its own knowledge and personal experiments.

Then the main research question is: **How does the concept of symmetry set up the organization and the inferences between the operational invariants relatives to**

the others transformations of the plan into the student's *personal GWS?* And how does this *personal GWS* evolute during secondary school?

METHODOLOGY

We propose a common test to students at two different levels: at the second year, after the teaching of the reflection through a point and, at the fourth year, after the teaching of the rotation. The students are 12-13 y.o. and 14-15 y.o. and have the same mathematics' teacher. We chose the situation of recognition of transformations because it is a usual task all along French secondary school. We define two different tasks from a same configuration with triangles but with different kind of graphical support. These tasks are given to students at two different times. The first task (Fig. 1) suggests a "Global Perception" (we will note GP) because triangles are indicated as a whole with numbers and the transformations are indicated with arrows. This does not mean the students are only involved on a global perception; they may use a punctual perception too. The terms of the problem are: *In each fallow case, indicate which reflection(s), translation(s), rotation(s) transform: a) 1→2 b) 2→3 and c) 1→4. Justify yours answers. If you add marks on the figure, please do not rub out.* The last question *c)* is only given to the students from the last year but we do not analysis the results because we are devoted to the case with reflection(s) and rotation(s). Furthermore, it is only indicated *which reflection(s)* (and not the other transformations) with the students from second year.

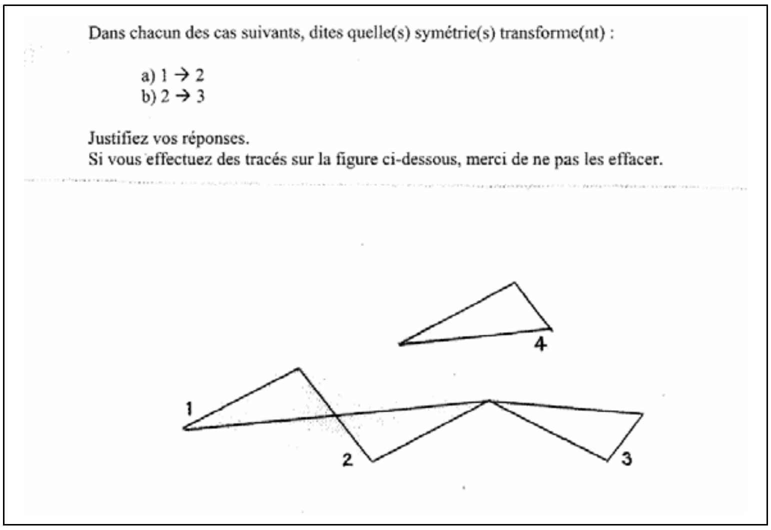
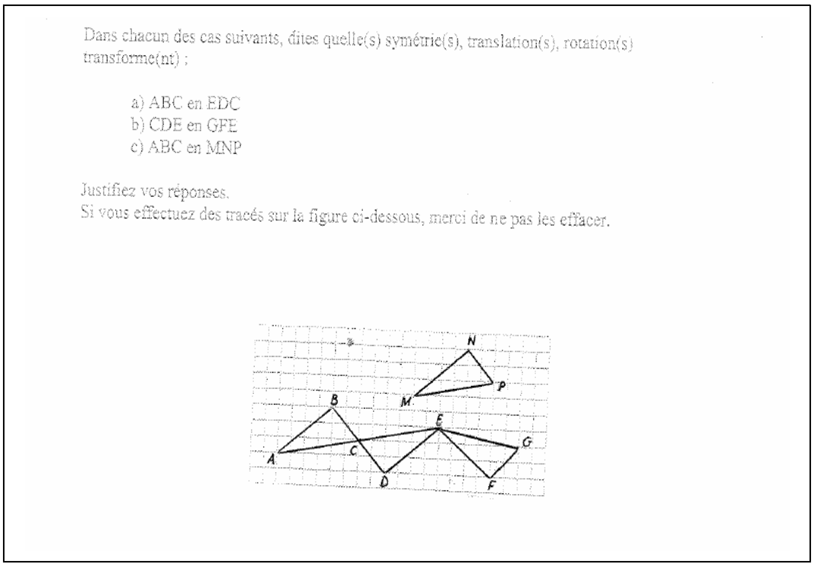


Fig 1: "The triangle situation" in the case called "Global Perception" (GP).

The second task, given one week later, is the same as previously but the terms of the problem suggest a "Punctual Perception" (we will note PP) to the students (Fig. 2). The configuration is given with a squaring and the triangles' tops are called by letters on the pattern and in the terms of the problem (*ABC in EDC*).



Dans chacun des cas suivants, dites quelle(s) symétrie(s), translation(s), rotation(s) transforme(nt) :

a) ABC en EDC
b) CDE en GFE
c) ABC en MNP

Justifiez vos réponses.
Si vous effectuez des tracés sur la figure ci-dessous, merci de ne pas les effacer.

Fig 2: "The triangle situation" in the case called "punctual perception".

These tasks are quite easy for these students (they have to recognize a reflection through a point or a rotation of 180° at the question *a)* and a reflection through an axis at the question *b)*). Different didactical variables are convened and then different students' strategies are implied in both tasks. In particular, the graphical support is different in both case, in the GP one, students' adaptations are wider: they may involve arguments based on superimposition (folding or half-turn) or build strategies based on metrics' arguments (Euclidian Affine Geometry) with measurement or perception. We suppose these latter strategies (with metrical arguments) are more effective in the task PP since there is a squaring and figures are nominated. Mathematical properties are not given as hypothesis in the term of the problems, so different types of metrical properties are acceptable (as for instance "AC=CE" or "AC and CE are almost equals" or even "AC is not equal to CE") but it is assumed a transformation has to be recognized. Moreover, the figural position is actually a didactical variable to consider and we should consider intermediate task (as for instance, without common point, etc.) in order to consolidate the results already got here. However, considering that, we show that students' behaviour changes according to the perception suggested by the task (as expected) but the adaptations imply a different way of figural treatment according to the transformation at stake and according to the students' grade. The aim of this paper is describe the differences between transformations and the influence from the concept of symmetry on these adaptations at these both levels at secondary school.

RESULTS AND DISCUSSION

Student's category according to stability of student's achievement

We collected 29x2=58 productions from students who are 14-15 y.o. and 26x2=52 productions from students who are 12-13 y.o. We classified students' productions according to the stability of their performance on both tasks, i.e. if student proposes a correct answer in the task GP and next if he changes or not his answer in the task called PP. We will write RIGHT (R) or WRONG (W) the student’s finale issue on these both tasks. Then, different profiles are exhibited according to the student’s achievement at the question a) (the correct transformation is the reflection through a point - or a rotation of 180°) and at the question b) (the correct transformation is a reflection through an axis). Finally, the main student’s profiles are presented on the table 3 and table 4, and count at least two students.

<table>
<tr><th colspan="2">Recognition of the reflection through a point (question a)</th><th colspan="2">Recognition of the reflection through an axis (question b)</th><th rowspan="2">Number of students</th><th rowspan="2">Indicative percentage of pupils
%</th></tr>
<tr><th>GP</th><th>PP</th><th>GP</th><th>PP</th></tr>
<tr><td>R</td><td>R</td><td>R</td><td>R</td><td>16</td><td>≈ 55</td></tr>
<tr><td>W</td><td>R</td><td>W</td><td>R</td><td>2</td><td>6,9</td></tr>
<tr><td>R</td><td>R</td><td>W</td><td>W</td><td>4</td><td>13,8</td></tr>
<tr><td>R</td><td>W</td><td>R</td><td>R</td><td>4</td><td>13,8</td></tr>
<tr><td colspan="4">At least one WRONG</td><td>10</td><td>≈ 34,5</td></tr>
</table>

Tab. 3: Student’s profile from the last year of secondary school (14-15 y.o) depending on whether student is successful.

<table>
<tr><th colspan="2">Recognition of the reflection through a point (question a.)</th><th colspan="2">Recognition of the reflection through an axis (question b.)</th><th rowspan="2">Number of students</th><th rowspan="2">Indicative percentage
%</th></tr>
<tr><th>GP</th><th>PP</th><th>GP</th><th>PP</th></tr>
<tr><td>R</td><td>R</td><td>R</td><td>R</td><td>9</td><td>≈ 34,7</td></tr>
<tr><td>R</td><td>R</td><td>W</td><td>W</td><td>3</td><td>11,6</td></tr>
<tr><td>W</td><td>W</td><td>W</td><td>W</td><td>3</td><td>11,6</td></tr>
<tr><td>R</td><td>W</td><td rowspan="2">W</td><td rowspan="2">W</td><td rowspan="2">4</td><td rowspan="2">15,4</td></tr>
<tr><td>W</td><td>R</td></tr>
<tr><td rowspan="2">R</td><td rowspan="2">R</td><td>R</td><td>W</td><td rowspan="2">3</td><td rowspan="2">11,6</td></tr>
<tr><td>W</td><td>R</td></tr>
<tr><td colspan="4">At least one WRONG</td><td>13</td><td>≈ 50</td></tr>
</table>

Tab. 4: Student’s profile from the second year of secondary school (12-13 y.o.) depending on whether student is successful.

According to these results, only 34,7 % students from the second year recognize both transformations with successful, whatever the perception suggested by the task; and only 55 % students among students from the last year of secondary school recognize both transformations with successful, whatever the perception suggested by the task.

The students' profiles from the second year are more fragmented than the students'ones from the last year. Therefore, we suppose the student's *Geometrical Working Space (GWS)* from the last year is more stabilized. What we need now is to determine what did each profile (especially what mistakes) and what kind of adaptations they made according to the perception and the transformation at stake.

Local analysis of the *Geometrical Working Space* through the figural treatment according to Duval's meaning

We analyse the *GWS* through its organization between the real space (marks on sheet of paper), the objects of reference from a mathematical model (Euclidian one), and the artefacts (tools, schemes). Inspired by Duval (2005), we focus on the way of treatment of the figure in order to describe these links into the *GWS*. Duval defines different kinds of "figural deconstruction". He opposes "instrumental deconstruction" which implies the use of tools to build the figure and "dimensional deconstruction" which implies links between figural units (for example the points A and B - dimension 0D - indicate the measure AB - dimension 1D) in order to exhibit mathematical properties. The latter deconstruction may imply a mathematical reasoning and suggests a geometrical paradigm closer to GII. Finally, we assume the fact the *GWS* is a favourable environment to analyse the process of conceptualization at stake because, according to Vergnaud's meaning, the notion of representation of the real world is at the heart of the process of conceptualization. Therefore, an analysis of students' productions in term of figural treatment (according to Duval's meaning) is a relevant way to describe the connection between the component of the *GWS* (Object of real world / tools / models of reference) and therefore allows us to approach the process of conceptualization at stake.

Results about students' productions at the end of secondary school (14-15 y.o.)

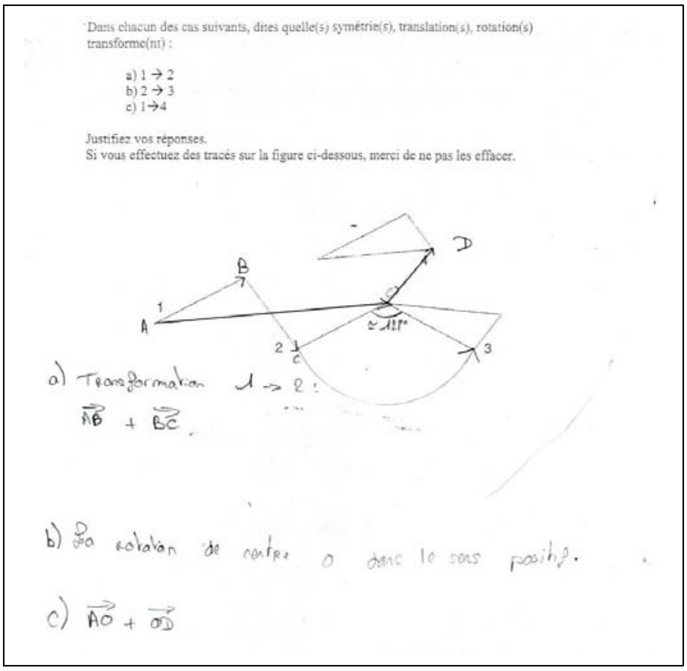
The student's *personal GWS* is adapted to the perception suggested by the task, as expected *a priori*. The operational invariants relative to the recognition of the reflection through an axis are different according to the task. The strategies of superimposition, folding or the use of common references are more present in the case GP than in the case PP.

Students may develop arguments from the Euclidian affine geometry with different kinds of "signifier" (Presmeg, 2006):

- signifier from an "instrumental deconstruction" (Duval, 2005), as for instance the *theorem-in-action of cocyclicity* : pupils use their compasses to test if a couple {point; image} of the figure belong to the same circle and therefore they infer it is a rotation. The language allows the denomination or describes the action.

- signifier from a “dimensional deconstruction” (Duval, 2005) through mathematical symbolism on the drawing (equality of measure, orthogonally, etc.). The language is used to announce the mathematical properties and make deduction.

These adaptations are used not only by students who propose correct answers but with students who propose wrong answers too. At the end of secondary school, we identify only one main kind of mistake made by students in these tasks. Students apply the *theorem-in-action of cocyclicity* at the question b) to recognize a rotation whereas it is actually a reflection through an axis (document 5).



Doc. 5: student’s production with a wrong use of the theorem-in-action of cocyclicity.

We suppose this mistake is from a “cognitive conflict” about the dimension of the mathematical objects at stake with different transformations (between rotation and symmetry). With this theorem-in-action, students do not control the conservation of the measure of the angle with other couples {point; image}. They only refer to an instrumental deconstruction and not to relevant mathematical properties to recognize a rotation. This mistake could be expected if we consider the relative position between triangles (with a common top) but in the case PP, the transformation is given point by point (“CDE in GFE”) and several cases show stronger relation with the figure (because they still use this theorem-in-action) whereas these same students may adapt their strategies according to the task if the recognition of reflection occurs (namely they use a dimensional deconstruction in order to refer to mathematical properties in the case PP). We have already noticed this mistake, called ‘theorem-in-action of cocyclicity” in a pre-test with others students with the same age (Bulf, 2007).

Results about students' productions at the second year of secondary school

If we compare the tab. 3 and tab. 4, students' profiles of 12-13 y.o. are more diversified. The personal *GWS* is still adapted to the perception suggested by the task but not as distinctly as for the students older, i.e. students use references to the real world mainly in the case GP but in the case PP too. On the other hand, they do refer to the Euclidian geometry in the case PP but sometimes in GP too. The mistakes are also more diversified because the adaptations to the perception suggested by the task are different than previously. We distinguish two main sorts of mistake:

- mistakes caused by "contract's effect" in the case PP. The notion of didactical "contract" is designed by Brousseau (1997) as a "relationship […] [which] determines - explicitly to some extent, but mainly implicitly - what each partner, the teacher and the student, will have the responsibility for managing and, in some way or other, be responsible to the other person for managing and, in some way or other, be responsible to the other person for. This system of reciprocal obligation resembles a contract". In our research, students propose mainly exhaustive explanations to solve the task in the case PP. They give too much mathematical properties to justify the transformation. Or, students change their mind and propose "institutional" properties on a wrong way to justify their choice in the case PP whereas their choice in the case GP was correct with naïve arguments from the real word. As for instance, one student justifies correctly the reflection through an axis (question b) in the case GP because he writes "it is possible to fold" but this same student writes, for the same transformation in the case PP, it is a reflection through a point because "in the reflection through a point, the image of a segment is a segment with the same length". This student proposes this same "argument" at the question a) too, but in this case it is coherent. This "institutional" sentence is exactly the same which is given during the classroom. This kind of mistake lets think that the "dimensional deconstruction" (he mentions segments) suggested by students' activity is artificial, and confirm Duval's point of view who pretend this cognitive operation is not self-evident.

- mistakes caused by "amalgam between notion on the same support" according to Artigue's meaning (Artigue, 1990). Students are confused with the reflection through a point and the reflection through an axis, because these both transformations imply the same schemes as for example the global superimposition, cutting in two both sides, the properties of equal distances, etc. In particular, some students recognize a reflection through an axis instead of a reflection through a point in the case GP (question a). Some other students recognize a reflection through a point instead of a reflection through an axis in the task called PP (question b). This kind of amalgam suggests the reflection through an axis is crystallized in a "global perception", at least at the beginning of secondary school.

CONCLUSION AND DISCUSSION

This research is devoted to the analysis of students' productions from two different levels at French secondary school. The students solved the same task given under two different forms (one is called "Global Perception" (GP) and the other one is called "Punctual Perception" (PP)). This research points out that the *personal Geometrical Working Space* is more stabilized for a student at the end of secondary school than for a student at the beginning of secondary school. The schemes of the concept of symmetry are more flexible and can be adapted to the task (arguments can be empirical or from deduction in the frame of Euclidian Affine Geometry according to the perception suggested by the task). These adaptations show a relevant expertise of the dialectic of paradigms GI-GII when the reflection through an axis is involved, for the older students. However, the analyses of the mistakes of these students show a difference of conceptualization between the rotation and symmetry. Rotation involves an "instrumental deconstruction" only, whereas the symmetry may involve "dimensional deconstruction".

The mistakes made by younger students imply a sort of amalgam between the different symmetries or imply the use of an artificial "dimensional deconstruction". These mistakes make unstable the *GWS* of these students.

This variation of the use and the effects of the concept of symmetry in the *personal Geometrical Working Space* leave questions about how is managed the concept of symmetry by the teacher during secondary school and how is managed the figural deconstruction. Duval has already mentioned the problem of transmission of the different crossing of figural deconstruction (2D, 1D, 0D) in classroom (Duval, 2005). He points out these different crossings are not so obvious for students, and the difficulty of these crossings are underestimated by teachers and curricula. This point concerns the rest of our research.

NOTES

1. Official instructions: http://eduscol.education.fr/. BO n°10 Hors-Série, 15/10/1998, pp. 106-114 (3^{e}'s instructions). BO n°5 Hors-Série, 09/09/04, pp. 4-16 (6^{e}'s instructions). BO n°5 Hors-Série, 25/08/2005, pp. 9-16 (5^{e}'s instructions).

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THE ROLE OF TEACHING IN THE DEVELOPMENT OF BASIC CONCEPTS IN GEOMETRY: HOW THE CONCEPT OF SIMILARITY AND INTUITIVE KNOWLEDGE AFFECT STUDENT' S PERCEPTION OF SIMILAR SHAPES

Mattheou Kallia *, Spyrou Panagiotis **

* Department of Education, University of Cyprus. Email: kallia99@yahoo.com

** Department of Mathematics, Athens University. Email: pspirou@cc.uoa.gr

ABSTRACT

In this research we investigate whether students of the Pedagogical Department of Education have the basic geometrical knowledge which is related mainly with the similarity of shapes. We also investigate how they define similarity of shapes and if the intuitive knowledge affects their perception of similar shapes. The results showed that students have developed certain structures in regard to some concepts in geometry based on the teaching that they have received in school. The results showed, as well, that a large percentage of students are not in a position to correctly define the similarity of shapes. Finally, research shown, that intuition affects their responses and their mathematical achievement.

INTRODUCTION

The role of geometry in the development of mathematical idea is very important. The geometrical skills and visual icons are basic instruments and source of inspiration for many mathematicians (Chazan & Yeryshalmy,1998 in Protopapas,2003). The content of geometry is appropriate both for the development of lower level of mathematical thinking, (i.e. the recognition of shape), as well as for higher order thinking, (i.e. the discovery of the properties of shapes), the construction of geometrical models and the solution of mathematical problems (NCTM, 1999). The representation of geometrical objects and the relationships between geometrical objects and their representations constitute important problems in geometry (Mesquita, 1998).

Geometry constitutes a basic part of the National Curriculum for Primary as well as Secondary Education. The concept of similarity between two shapes is taught in the 3rd grade in Secondary School and in the 1st grade in higher Secondary School, with special emphasis on the similarity of triangles. The teaching mainly concerns, understanding of the concept of similar shapes, i.e. that similar shapes are those which their sides are proportional and their angles that are created by the respective angles are equal.

Literature review has shown the concept of similarity is presented and taught through the environment of dynamic geometry and mainly through the use of applets. The concept is taught in coordination to the teaching of symmetry and transformations that can occur in a shape (http://standards.nctm.org/document/eexamples/chap6/6.4).

In addition, the properties of similar shapes are presented and in the proof of Thalis theorem. This theorem has some applications and proofs with the use of the Geometer Sketchpad. Although there are no relationships presented in regard to the results and consequences (proportion of relationships of line segments) of Thalis Theorem and the concept of similarity of shapes (beyond quadrilaterals).

The common teaching environment of geometry is very limited in formal education. For example, the constructions that the children are asked to deal with, the shapes are placed in a horizontal position, i.e. the sides are parallel to the sides of the object on which the construction is done. As a result most students develop an holistic and stereotype view of the geometrical shapes which is very affected by the intuitive rules.

At the university level, the students of the Department of Education are taught geometry through its historic evolution. In order to be able to follow and understand these lectures basic knowledge of geometry is required. This knowledge is mainly provided at the 3rd year of secondary school. Unfortunately, students appear to be lacking knowledge. This may be due to the long interval that has transpired since they dealt with geometry or due to the teaching in higher secondary school where it is mainly expected by the student to memorize relationships instead of understanding and applying them.

It is possible that the level of mathematical thinking may be influenced by some factors which are mathematics specific, such as the specific mathematical terminology which may be in conflict with the meaning we give to these terms in every day life or the conclusions that we reach based on the intuitive view of mathematical knowledge.

The aim of the present study is to investigate whether the students participating in EPA 171 (Basic concepts in mathematics) have the basic geometrical knowledge that is required for this specific course. It aims to investigate students' knowledge in regard to the similarity of shapes and how their intuitive knowledge may affect their perceptions about similar shapes.

THEORETICAL BACKGROUND

Geometry is comprised by three kinds of cognitive procedures which carry out specific epistemological functions (Duval, 1998):

a) Visualization: Is the procedure which is related to the representation of space in order to explain a verbal comment, for the investigation of more complex situations and for a more holistic view of space and subjective confirmation.

b) Construction with the use of apparatus. The construction of shapes can act as a model.

c) Reasoning: Is investigated in relation to verbal procedures and the extension of knowledge for proof and explanation.

These different procedures can be carried out separately. Thus the visualization is not based on the construction. There is however access on the shapes and the way that they have been constructed. Even if the construction leads to visualization, the construction is based only on the connections between mathematical properties and technical restriction of the apparatus which are used. Furthermore although the visualization is an intuitive aid, necessary in is some instances for the development of proof, still the justification is solely depended on a group of sentences (definitions, axioms, theorems) which are available. In addition to this visualization is sometimes more deceptive or impossible. Still these three kinds of cognitive procedures are closely linked and their cooperation is necessary for any progress in geometry (Protopapas, 2003).

The concept of similarity:

Similarity constitutes a basic link between algebra and geometry and it also has a close relationship to trigonometry. The theorem which expresses that two similar triangles have their sides proportional and Pythagoras theorem constitute two basic links between geometry and algebra. The connection of geometry and algebra is particularly construction as it allows using the visualization of geometry in algebraic problems and the flexibility of algebraic operations in geometrical problems. Similar triangles and the Pythagoras theorem constitute the cornerstone of Trigonometry. By using similar triangles we can calculate the sides and angles of an object by measuring the lengths of a smaller model.

According to Vollrath (1977) in geometry similarity constitutes a relationship between shapes/figures. A shape F1 is similar to a shape F2 if there is a transformation s such as $s(F_1) = F_2$. i.e. the square is similar to another one only when the ration of their sides is the same. In a didactical situation this constitutes a conclusion. Similar conclusions may be reached in regard to triangles and polygons. The proof is given based on the definition, using the properties of similar transformation. For a spiral approach of geometry it is important to know when it is possible to extract conclusions in regard to the understanding of similarity as it is defined through geometry or based on everyday language before teaching definition. Nevertheless, students do not seem to use the idea of sides' proportion to secure an exact answer about the similarity of shapes in enlargement or deduction in size of a shape (Kospentaris and Spyrou, 2005).

This can form the basis for a general definition of the concept of similarity. For the teaching of similarity at University level it is necessary, the lecturers to know in what extent the link between representation and expression of the concept of similarity can support or inhibit the cognitive procedure for this relationship. Furthermore it is important to know the explanation that the students give to similarity as it is used in everyday life or in a geometrical model (Vollrath, 1977). Kospentaris and Spyrou (2005) confirms in their study that the term similarity in everyday language does not in any way coincide with geometrical similarity, being more close to the meaning of having the same size.

The understanding of the concepts of similarity can be tested with exercises of classifying geometrical objects due to the fact that similarity constitutes a relationship of similarity between shapes/figures. In the teaching of mathematical the exercises of classification direct students in the study of properties and the properties that characterize concept and lead them to the extraction of definitions and they coordinate the understanding of definitions. Due to their importance we use exercises on classification to investigate students' understanding related to similarity irrespective of the mathematical definition. (Vollrath, 1977).

Intuition – and how it affects the teaching in mathematics:

As suggested by Fischbein (1999) intuition constitutes a theme that mostly philosophers are interested in. According to Descartes (1967) and Spinoza (1967) intuition appears to be a genuine source of pure knowledge. Kant (1980) describes intuition as the ability which leads directly to your goals and indirectly to the basic knowledge. Bergson (1954) made a distinction between intelligence and intuition. Intelligence is the way in which one may know the physical world, the world of stability, the extent of the properties of statistical phenomena. Through intuition we have a direct perception of the essence of spiritual life and control of the phenomena, time and motion (Fischbein, 1999).

Some philosophers, such as Hans Hahn (1956) and Burge (1968), have criticized intuition and its effect, in its scientific explanation. They believe that intuition leads to deceptive results and this has to be avoided in the scientific procedure.

The investigation of intuitive knowledge appears mainly in the work of people that are interested in scientific and mathematical understanding of students (for example Clement et al., 1989; DiSessa, 1988; Gelman and Gallistel, 1978; McCloskey et al., 1983; Resnick, 1987; Stavy and Tirosh, 1996; Tirosh, 1991 in Sierpinska, 2000). There is not an accepted definition of intuitive knowledge. The term: "intuition" is used mainly as a mathematical basic term such as the point or line (Sierpinska, 2000).

The importance of definition is probably respected just like the elements that are based on intuition. The basic common properties of these are based on individual proofs which are in conflict to logical and analytic attempts.

The problem of intuitive knowledge has earned an important place in scientific attempts. On one hand scientists need intuition in their attempt to discover new strategies, new theoretical and empirical models and on the other hand they need to be acquainted with what does not constitutes intuitionv– according to Descartes and Spinoza – basic guarantee, fundamental basis for objective truth.

The interest in regard to intuition also stems from the teaching of science and mathematics. When you need to teach a chapter in science or mathematics you often discover that what was already a fact for you – after university level studies – comes in conflict with basic cognitive obstacles that the students exhibit in their understanding. As a teacher you often believe that students are ready to memorize what they have been taught, actually they understand and memories relative knowledge. Intuitive perception of phenomena is often different that to their scientific explanation.

In mathematics, the belief that a square is a parallelogram is intuitively very strange for many children. The belief that by multiplying two numbers we may get a result that is smaller than one or both the numbers which we have multiplied is also difficult to be accepted. Intuition affects many of our perceptions. The educator discovers that the knowledge which s/he is supposed to transfer to the students is in conflict, very often, with the beliefs and explanations which are direct and solid and at the same time in conflict with the scientifically accepted perceptions.

THE STUDY

Aim:

The aim of the study is to investigate whether the students participating in EPA171 (Basic concepts in mathematics) have the basic geometrical knowledge that is related mainly with the similarity of shapes. How do they perceive the concept of similarity of shapes and how their intuitive knowledge may affect their understanding of similarity of shapes?

The three hypothesis of the study were:

1. The students have specific difficulties in basic concepts in geometry.
2. The students define similarity of shapes based on similar triangles or intuitive knowledge.
3. Intuitive knowledge affects their perception of similar shapes.

Subjects:

The participants in this study were 85 students of the Pedagogical Department of Education. 42 had mathematics as a major subject in higher secondary school, 39 had mathematics as a core subject and 4 did not specify.

Design of the study:

In order to examine the hypothesis of this study a test was administered to all the students that took part in the study. The students had 40 minutes available to respond to the test. The tasks of the tests were related with basic geometrical concepts (definition and construction of obtuse angle, application of properties of parallel lines and of the Pythagoras theorem in the solution of relevant exercises), definition of similarity of shapes, recognition of similar shapes as well as tasks which were used to examine whether the students had the necessary knowledge which is required to teach the lesson.

For the analysis of the results descriptive statistic as well as the implicative analysis have been used. More specifically for the data analysis the following elements of implicative analysis have been utilized: (a) The similarity tree-diagram which shows the variables according to the similarity they show (b) the hierarchical tree-diagram which presents the implicative relationships according to the order of significance.

Results:

The first hypothesis is confirmed in that basic knowledge of geometry where no special attention is given in school, such as the ability to give the definition of concepts. For the examination of this hypothesis which concerns basic geometrical concepts four questions were posed.

The first two questions were related mainly to the mathematical terminology which the students use. Students were asked to give a definition and construct an acute angle and it's supplementary. The analysis of the results shows that 83% can draw an obtuse angle but they only refer to the fact that it has to be bigger than 90^o but they do not specify that it has to be smaller than 180^o. 14% of the students who are mostly the ones that had mathematics as a major subject in higher secondary give a complete answer, whereas 3% of the students can not answer this basic question at all. In regard to the question related to the supplementary angles 95% give a complete answer since only one condition is requested (sum 180^o) and only 5% does not answer or gives a wrong answer.

The third question of the test concerns the use of basic relationship between angles and is based on parallel lines and the solution of a problem. These relationships are used quite extensively in secondary education something that leads students to a direct recognition and use of the relationships. This is illustrated by the results in the test since the majority (90%) that dealt with the task in question 3 managed to give correct answers.

The forth question of the test require a direct application of Pythagoras theorem twice. The application of Pythagoras's theorem without its proof constitutes a basic chapter in the teaching of geometry in secondary school. Thus 82,5% of the students were able to solve the exercise, 4,5% were able to solve only half of the task and 13% either gave a wrong answer or did not provide a response.

The second hypothesis was not fully confirmed. More than a third of the students could give a complete answer and a significant percentage of students referred to the similarity of the appearance of the shapes or the similarity of triangles. In order to examine this hypothesis the questions 5a and 5b were given.

In the question 5a, which asked students to answer "what are similar shapes?" only 36,5% of the students were able to give a complete answer (5iv). 21% referred to the similarity in the appearance of the shapes (5iii) and 14% referred to the similarity of triangles (5ii) which plays a significant role in the teaching of similarity in secondary education. A significant percentage of the students 12% referred to equality (5i), whereas 16% of the students either did not provide any answer or gave a wrong response (5i).

In order to examine whether the students have the ability to use the definition of similarity of shapes in an exercise regarding similar triangles, the second part (5b) of exercise 5 was asking students to find the relationship of similarity between given triangles. Differently to their responses in the 1st part of the exercise where 53% could give a complete answer, only 30% were able to reach a mid way to the solution. 17% could not solve the problem or did not give any response.

For the application of the theory regarding the relationships of similarity and also for the examination of the third hypothesis, exercise 8 was presented where students were asked to find which polygons are similar. In contrast to exercise 5b where they had to write some relationships algebraically in order to prove the similarity of the shapes, in this exercise, they needed mental representations of the relationships so that the right choices could be made. Just like in question 5, some students confuse similarity with the relationships regarding the appearance of the shape. That is probably why 87% responded that the parallelograms that have equal angles one side proportional and one side equal are similar (8i). It is very likely that they have reached this answer because both of them are parallelograms. 13% of the students believe that the rectangles are similar to the square (8iv) in the shape. This may be due to the fact that all three of them are parallelograms (appearance of the shape). Similarly 6% believe that the right angle triangle is similar to the scalene triangle (8v), most probably because both of the triangles have the same appearance. 80% recognize the similarity of the rectangles that are presented (8iii) and of the right

Question 5: What are similar shapes?
5i: referred to equality or no answer or wrong response.
5ii: referred to the similarity of triangles
5iii: referred to the similarity in the appearance of the shapes
5iv: complete answer

Exercise 8: students were asked to find which polygons are similar.
8i: responded that the parallelograms that have equal angles one side proportional and one side equal are similar.
8ii: recognize the similarity of the right angle triangles
8iii: recognize the similarity of the rectangles
8iv: believe that the rectangles are similar to the square in the shape.
8v: believe that the right angle triangle is similar to the scalene triangle

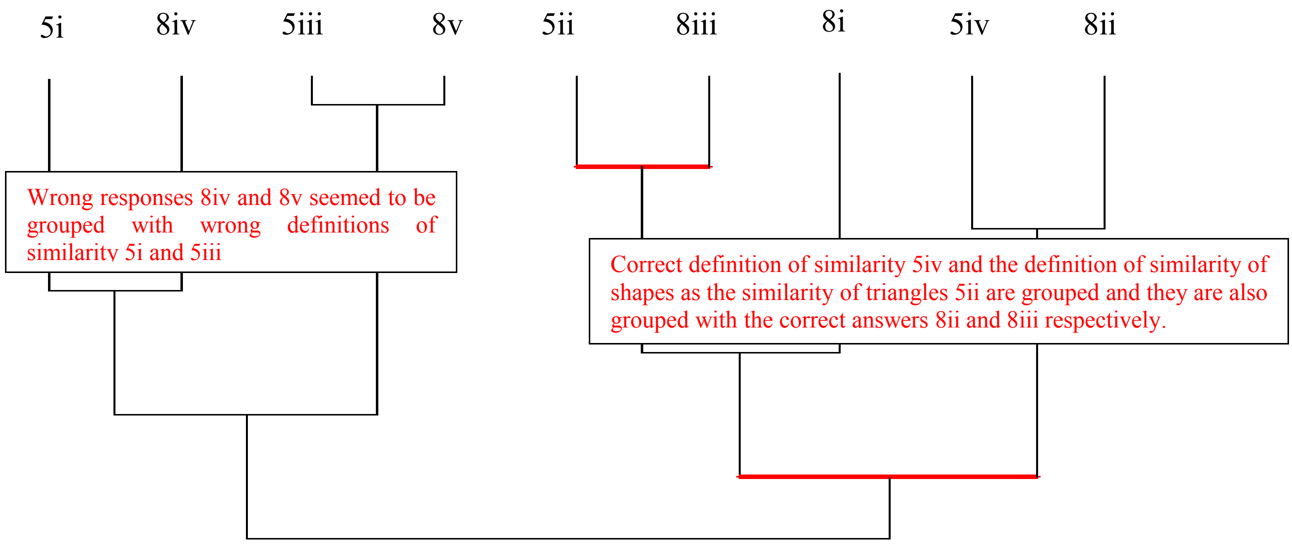

Figure 1: similarity tree diagram

In order to examine whether the definition that students give for the similarity of shapes affects their answer in exercise 8 where they are asked to recognize similar shapes we have used the similarity tree diagram (Figure 1). In the tree diagram the wrong responses in exercise 8 seemed to be grouped with the variables 8iv and 8v (similar shapes: square-rectangle, variable 8iv and right angle triangle and scalene triangle 8v) with the variables 5i and 5iii respectively of exercise 5 which refer to wrong definitions of similarity (5i: equality of shapes or wrong answer and 5iii: similarity in the appearance of the shape). In addition to this, the correct definition of similarity (variable 5iv) and the definition of similarity of shapes as the similarity of triangles (variable 5ii) are grouped and they are also grouped with the correct answers in exercise 8, and the variables 8ii and 8iii respectively. The variable 8i which is the wrong answer in 8 since it presents the similarity of two parallelograms that their sides are not proportional appear to be grouped with the correct definitions (mainly with the definition of similar triangles and the correct answer in regard to rectangles) and the correct answers. This may be due to the fact that most students perceive as the correct answer, something that indicates that students are depending on the perception of shapes and not the definitions and the properties of the shapes.

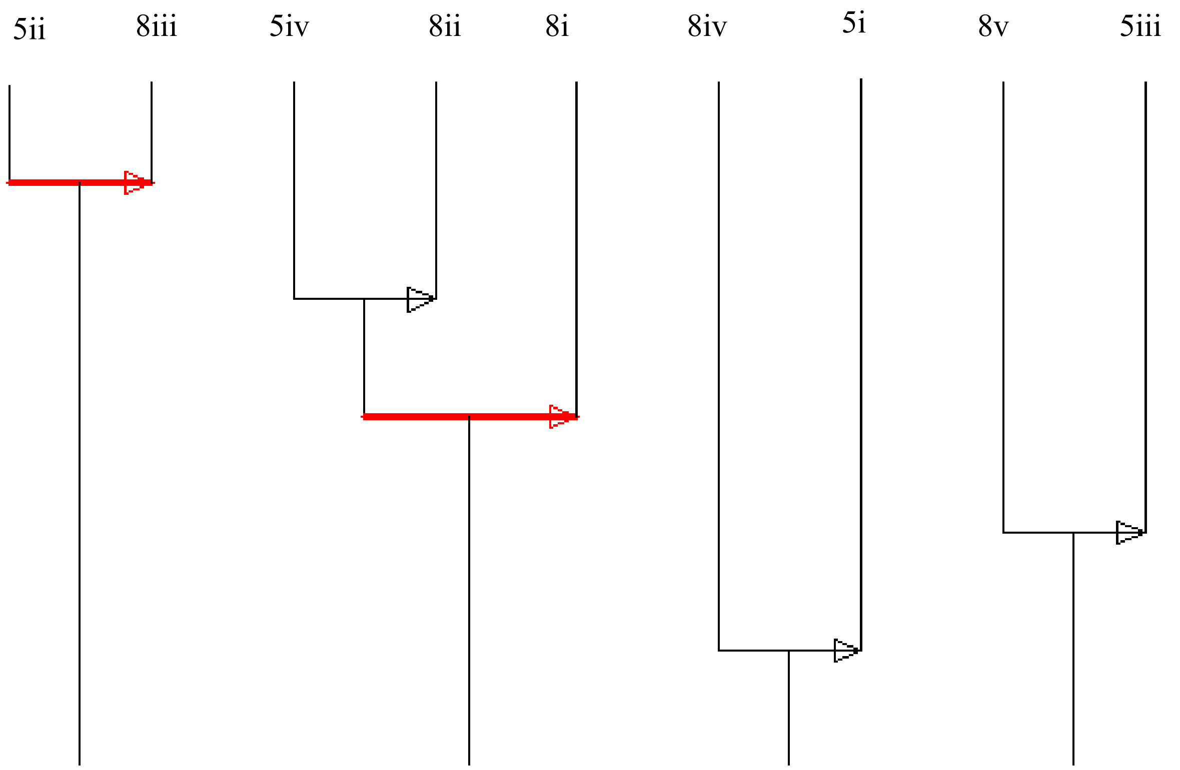

Figure 2: hierarchical diagram

The hierarchical diagram (Figure 2) shows that success in the definition constitutes success in the tasks in exercise 8, whereas in the wrong responses higher in line are the tasks in exercise 8, something that results to difficulty in giving a correct definition for the similarity concept.

CONCLUSIONS

The data of the study suggest that students have developed certain structures in regard to some concepts in geometry based on the teaching that they have received in school. The fact that in secondary education more emphasis is placed on the practical application of theory and less on the understanding of concept, leads to students´ difficulty in giving complete definitions that require conditions, which in the practical application are implied without being presented (for example, the representation of an obtuse angle is never presented opposite to angles bigger than 180^o and that is why students never refer to the condition that an obtuse angle needs to be smaller than 180^o).

Based on this it appears that students are in a position to carry out operations by using certain formulas (Pythagoras's theorem) or recognize relationships in shapes which they were taught in school and they are expected to apply these in exercises similar to exercises 3 and 4 of this test.

For a spiral approach and development of geometry, it is important to know when it is possible to extract conclusions in regard to the concept of similarity as it is defined in geometry. As it appears from the data, a large percentage of students are not in a position to correctly define the similarity of shapes. However they are able to apply the relationships of similarity in triangles since teaching in secondary education is related to the similarity of triangles

In the search for similarity relationships in exercise 8 students influenced by their intuition found relationships that were based on the similarity of the appearance of the shape but they were not mathematically similar. This indicates that intuition affects their responses and their mathematical achievement since a number of these students have not received adequate mathematical training in order to base their answers on definitions, properties of the shapes and not on the perceptual appearance of the shape.

The data suggest that the wrong similarity definition leads to wrong responses in the practical applications, whereas the wrong representations of concepts create students' erroneous structures and definitions of the specific concepts.

In conclusion, in regard to the teaching of geometry at University level it is important to give more attention in the teaching of basic geometrical concepts and skills. As it was shown by the results in this study the teaching that many students receive in secondary school is inadequate, something that affects their perception and achievement in geometry. The lack or limited knowledge that students have, lead, to the use and translation of mathematical definitions based on wrong mental representations which are affected by intuitive knowledge and not by the correct mathematical definitions and correct representations.

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THE GEOMETRICAL REASONING OF PRIMARY AND SECONDARY SCHOOL STUDENTS

Georgia Panaoura and Athanasios Gagatsis

University of Cyprus, Department of Education

In the present paper comparing the geometrical reasoning of primary and secondary school students was mainly based on the way students confronted and solved specific geometrical tasks: the strategies they used and the common errors appearing in their solutions. This comparison shed light to students' difficulties and phenomena related to the transition from Natural Geometry (the objects of this paradigm of geometry are material objects) to Natural Axiomatic Geometry (definitions and axioms are necessary to create the objects in this paradigm of geometry) and to the inconsistency of the didactical contract implied in primary and secondary school education. These findings stress the need for helping students progressively move from the geometry of observation to the geometry of deduction.

INTRODUCTION

Teaching geometry so that students learn it meaningfully requires an understanding of how students construct their knowledge of various geometric topics (Battista, 1999). This means it is necessary that mathematics educators investigate and mathematics teachers understand how students construct geometrical knowledge as a result of their learning experiences in school. An important aspect of this research direction is the study of the strategies that students use in different geometrical tasks as well as the identification of their mistakes. In the work of Piaget and in the Geneva School we see that errors were for the first time viewed positively, in the sense that they allow the tracing of the reasoning mechanisms adopted by students.

The literature review reveals that the investigation of various issues related to students' geometrical reasoning (knowledge, abilities, strategies, difficulties) is in most cases restricted to the study of groups that come from one educational level. We believe that it is necessary to gather empirical data which would allow the comparison between groups of students in primary and secondary education and would be valuable sources of information regarding aspects of teaching in the two educational levels as well as the difficulties met by students of different age groups.

The transition from elementary to secondary school is recognized as a critical life event, since, progressing from one level of education to the next, students may experience major changes in school climate, educational practices, and social structures (Rice, 2001). Research results reveal substantial agreement that there is often a decline in students' achievement following this transition, but achievement scores tend to recover in the year following the transition (Alspaugh, 1998). In the case of Cyprus, students experience difficulty during the transition from elementary to secondary school which is evident in their performance in most topics, especially in mathematics.

This paper is based upon a research project which investigated the transition from elementary to secondary school geometry in Cyprus, gathering data concerning students' performance in tasks involving two-dimensional geometrical figures, three-dimensional geometrical figures and net-representations of geometrical solids, as well as the students' spatial abilities. In the present paper we focus on the strategies the students used to solve specific geometrical tasks involving two-dimensional figures and we study the kinds of errors that we identified in the students' solutions.

THEORETICAL BACKGROUND

In the present paper we use as explanatory framework Duval's cognitive approach to geometry (Duval, 1995, 1998) and the framework of Geometrical Paradigms proposed by Houdement and Kuzniak (Houdement & Kuzniak, 2003; Houdement, 2007). We also use the concept of the didactical contract, introduced by Brousseau (1984) to interpret some of the students' wrong answers. According to him, the didactical contract is defined as a system of reciprocal expectancies between teacher and pupils, concerning mathematical knowledge. The didactical contract is in large part implicit and is established by the teacher in her teaching practice. The students may interpret the situation put before them and the questions asked to them on the basis of the didactical contract and act accordingly.

A cognitive approach to geometry

Duval (1998) argues that geometry involves three kinds of different cognitive processes – visualization processes, construction processes and reasoning in relation to discursive processes – the synergy of which is necessary for proficiency in geometry. Approaching geometry from a cognitive point of view, he has distinguished four cognitive apprehensions connected to the way a person looks at the drawing of a geometrical figure: perceptual, sequential, discursive and operative (Duval, 1995). Briefly, perceptual apprehension refers to what a person recognizes at first glance when looking at a geometrical figure, while sequential apprehension is required whenever the construction or description of construction of a figure is involved. Discursive apprehension refers to the mathematical properties that cannot be determined through perceptual apprehension of a figure, but must be given through speech or can be derived from the given properties. Operative apprehension depends on the various ways of modifying a given figure. Solving geometrical problems often requires the interactions of these different apprehensions, and "what is called a 'geometrical figure' always associates both discursive and visual representations, even if only one of them can be explicitly highlighted according to the mathematical activity that is required" (Duval, 2006, p.108).

The framework of Geometrical Paradigms

Keeping the idea of 'paradigm' from Kuhn, who used it to explain the development of science, Houdement and Kuzniak (2003) proposed that elementary geometry appears to be split into three various paradigms, characterizing different forms of geometry: Geometry 1 (natural geometry), Geometry 2 (natural axiomatic geometry) and Geometry 3 (formalist axiomatic geometry). The theoretical framework they have developed specifies the nature of the geometrical objects, the use of different techniques and the validation mode accepted in each of the three paradigms. Here we briefly describe the first two geometrical paradigms distinguished by Houdement and Kuzniak (Houdement & Kuzniak, 2003; Houdement, 2007), which mainly concern primary and secondary school students that participated in the present study.

Geometry 1 is intimately related to reality and reasoning is close to experience and intuition. The objects of Geometry 1 are material objects, graphic lines on a paper sheet or virtual lines on a computer screen. Drawing and measurement techniques with ordinary geometrical tools (ruler, set square, compass) as well as experimentation in the sensible world (using techniques such as folding, superposing) are used in this paradigm. New knowledge may be produced based on evidence, experience or reasoning, while a permanent motion between the model and the reality enables the student to 'prove' the assertions.

In Geometry 2 the objects are ideal, so reasoning relies on the mathematical properties of the abstract geometrical objects. A system of definitions and axioms is necessary for the creation of the objects. In this system the axioms are as close as possible to intuition, but making progress and reaching certainty demands demonstrations inside the system. Hypothetical deductive laws are the source of validation.

THE PRESENT STUDY

As noted in the introduction, this paper is based upon a research project which examined primary and secondary school students' geometrical knowledge and abilities related to tasks involving different geometrical figures, as well as their spatial abilities in micro-space. Participants in our study were 1000 primary and secondary school students (488 males and 512 females) from 29 classes of 9 elementary schools and 12 classes of 8 secondary schools in four different districts of Cyprus. Specifically, the sample involved students from three grades (fourth grade – primary school: 332, sixth grade – primary school: 333 and, eighth grade – second grade of secondary school: 335). The mean age of the three grades was as follows: fourth grade, 9.8 years; sixth grade, 11.7 years; eighth grade, 13.9 years. Information concerning the instrument we constructed for the purpose of our research project and the procedure we followed can be found in Panaoura and Gagatsis (2008).

In the present paper we attempt to compare the geometrical reasoning of primary and secondary school students (the three age groups in our study) based on their solutions to three specific geometrical tasks which involved two-dimensional figures (the three tasks are shown in the Appendix). At this point we have to stress that the comparison attempted here does not refer to the levels of success of the three groups of students, since we study students of different age, from different educational levels, with different learning experiences and different cognitive abilities. Using as explanatory framework the theoretical notions presented above, we focus on the strategies and the common errors we identified in students' solutions. In this direction first we present part of the results from our study concerning students' solutions of three geometrical items included in the test and then we discuss these results and students' difficulties under the light of didactic phenomena rising from our research.

RESULTS ON SPECIFIC GEOMETRICAL ITEMS

Item [A]

On the geometrical figure presented in item [A] a square and a right triangle can be identified. In order to give the correct answer, the students had to (a) identify, within the figure presented, the subfigures of the square and the right triangle, (b) pass from 2D to 1D and 'see' that the unknown segment [AC] is one of the square's sides and (c) recall and apply the cognitive unit referring to the property of equal sides in a square. At this point we must note that in the geometry test we included a multiple choice item to examine whether students possess the cognitive unit referring to the property of equal sides in a square. The results presented in Table 1 showed that while a high percentage of the students answered correctly to the specific multiple choice item (61.7% of 4th graders, 85.9% of 6th graders and 86.9% of 8th graders) – indicating they *know* that the four sides of a square are equal – a smaller number of students (especially from primary school) eventually gave a correct answer to the geometrical item [A].

Item	Answer	4th graders	6th graders	8th graders
Multiple choice	Correct	61.7	85.9	86.9
Item [A]	Correct – using properties	36.4	71.8	66.9
	Correct – applying theorem	---	---	18.5
	Wrong – using ruler	8.4	2.1	---
	Wrong – arithmetical operations	6.0	4.8	2.4

Table 1: Students' answers to multiple choice item and item [A] by age group

Crosstabs tables of performance to the multiple choice item by performance to item [A] were obtained for each age group in order to examine what percentage of the students who answered correctly to the specific multiple choice item, did actually solve the geometrical item [A]. The crosstabs results indicated that half of the 4th

grade students and a percentage of 22% of the 6^{th} grade students who gave the correct answer to the multiple choice item (know that the sides of a square are equal) were not able to produce a correct answer to item [A]. The corresponding percentage was 10% in the case of 8^{th} grade students. So it seems that the secondary school students, working in the Natural Axiomatic Geometry paradigm, generally felt the need to use the properties and recalled the right one to solve item [A].

On the other hand, examining at the common errors identified in the students' solutions (Table 1), we notice some primary school students who gave (wrong) answers after using their ruler to measure the unknown segment on the geometrical figure presented on their paper. Additionally, a small number of students of the three age groups tried to combine the arithmetical data of the problem in a random way in arithmetical operations in order to come to an answer.

At this point it is interesting to state that, while the students could give the correct answer to item [A] by simply applying the property of equal sides in a square, we identified 18.5% of the secondary school students who solved the specific geometrical problem by applying Pythagoras' theorem in the subfigure of the right triangle. This performance is probably influenced by a part of the didactical contract according to which they are expected to apply Pythagoras' theorem any time a right triangle is involved in a geometrical figure. On the other hand, the specific performance indicates a difficulty concerning the transition from primary to secondary school. Specifically, the emphasis put on the use of algorithms during mathematics teaching in the secondary school seems to gradually result to the phenomenon that the students feel the safe of using an algorithm to be greater than that of a simple application of a geometrical property.

Items [B] and [C]

In Table 2 we present the results of students' attempts to solve two other geometrical tasks included in our test (item B and item C). Item [B] is a problem given to French students entering middle school (Duval, 2006). Item [C] was constructed for the present study, as an analogous problem to item [B], with two basic differences. First, on the geometrical figure presented in item [B], the subfigures of a circle and a rectangle appear, while on the geometrical figure presented in item [C] the two subfigures identified are a square and a rectangle. Second, the 'visibility' of the geometrical figure (and its subfigures) is less in the case of item [B] due to the specific configuration.

Facing the geometrical problem presented in item [B] a number of students in the present study relied only on a visual perception of the figure (perceptual apprehension) and either considered point E as the middle of [AB] (16.5% of 6^{th} grade students and 9.3% of 8^{th} grade students), or answered that the length of segment [EB] is equal to the circle's ray, "because *it seems to be* equal to the ray" (11.1% of 6^{th} grade students and 9.0% of 8^{th} grade students).

	Item [B]			Item [C]		
Answer	4th graders	6th graders	8th graders	4th graders	6th graders	8th graders
Correct – using properties	15.1	33.3	51.9	46.1	62.2	81.5
Wrong – visual perception (i)	6.6	16.5	9.3	3.3	4.5	0.6
Wrong - visual perception (ii)	8.7	11.1	9.0			
Wrong – using algorithms	10.2	5.4	0.9	11.4	9.9	2.1

Table 2: Students' answers to item [B] and item [C] by age group

In order to solve the item [C], the solver had to identify the two subfigures, to possess and to use the cognitive unit referring to the property of equal sides of a square. As in the case of item [B], a number of students relied only on the visual perception of the given figure and considering point E as the middle of [AB] answered that the length of segment [EB] is equal to 3.5 cm. In both cases perceived features of the geometric figures (relying on a perceptual apprehension of the given figure in each problem) have misled the students as to the mathematical properties involved in the problem solution and have obstructed appreciation of the need for discursive apprehension of the presented geometrical figure.

Finally, it is interesting to note that, as in the case of item [A], there are (mainly primary school) students who tried to give an answer to the items [B] and [C] combining in arithmetical operations the data presented in the geometrical problems. A possible explanation to the specific students' performance is that, according to the implicit didactical contract (Brousseau, 1984) established during the teaching and learning processes in the mathematics classroom – especially the aspect concerning the solution of routine arithmetical word problems – when those students are given a geometrical problem which involves arithmetical data, they suppose that they are expected to combine them in order to give an answer. They probably consider that in this way not only they can give an answer, but they also demonstrate that they have tried to solve the problem by identifying and using the data given in the problem. So, they assume that their teacher will be pleased with their performance!

DISCUSSION

Research about the learning of mathematics and its difficulties "must be based on what students do really by themselves, on their productions, on their voices" (Duval, 2006, p. 104). In this paper we presented some results from our research referring to the solutions of primary and secondary school students in three geometrical items, focusing on the strategies they used and their common errors. Once again we stress that we did not seek to compare students' levels of success, since it is obvious that the students participating in our study have different learning experiences (as far as the amount of experiences and the teaching methods are concerned) and differ in their cognitive development. The comparison of the solutions of the different age groups students shed light to phenomena related to the transition from Natural Geometry to Natural Axiomatic Geometry and to the inconsistency of the didactical contract implied in primary and secondary school education.

The transition from Natural Geometry to Natural Axiomatic Geometry

The passage from Geometry 1 to Geometry 2 is a complex, sensitive and crucial matter (Houdement & Kuzniak, 2003), since these two paradigms are different as far as objects, techniques and validation mode are concerned (Houdement, 2007). Moving from Natural Geometry to Natural Axiomatic Geometry students have to change their theory concerning the nature of the objects and of the space. They are forced to adopt the notion of conceptual objects, the existence of which is based on a definition in an axiomatic system. Consequently, they have to foster new techniques to work relying on the mathematical properties of each abstract geometrical figure.

The findings of the present study indicate that students working in the paradigm of Natural Geometry (mainly primary school students in our study) tend to consider geometrical objects as material objects and specific pictures rather than as theoretical, ideal objects which bear specific properties. This difficulty results to the phenomenon of students trying to solve geometrical problems often relying on the visual perception of the given geometrical figure rather on a mathematical deduction based on the properties of the geometrical objects involved. This phenomenon is related to the students' difficulty to work with geometrical figures as 'figural concepts' (Fischbein, 1993). We call it 'geometrical figure to figural concept' difficulty. As Mariotti (1995) has noted, correct and effective geometrical reasoning is characterized by the interaction and the harmony between figural and conceptual aspects of geometrical entities. In the present study, students working in the Natural Geometry paradigm (mainly primary school students) base their geometrical reasoning on the perceptual apprehension of the geometrical figure presented in a given task and this results to erroneous solutions, since the geometrical properties cannot be determined only through the specific type of apprehension. The perceptual apprehension of a geometrical figure must be under the control of the verbal propositions (discursive apprehension) which are presented in a geometrical problem (Duval, 1998), in such a way that correct geometrical reasoning results through the combination and interaction of the verbal propositions and the geometrical figure. In contrast to the students working under the Natural Geometry paradigm, students working in the Natural Axiomatic Geometry paradigm (mainly amongst secondary school students) focus their efforts on geometrical relations and they confront geometrical tasks based on the properties of geometrical figures (Houdement & Kuzniak, 2003).

Inconsistency of the didactical contract in primary and secondary education

The strategies used by the students in the solution of the presented tasks indicate that the didactical contract which is established among teachers and students concerning geometry learning in primary school education does not discourage all the students from (a) extracting conclusions based on the visual perception of a geometrical figure and (b) giving an answer extracted from random combination of the arithmetical data given in a geometrical problem. These aspects of the didactical contract were not identified to be present in the secondary school education, in the Natural Axiomatic Geometry paradigm, where the emphasis is on the properties of geometrical objects. We call this phenomenon "inconsistency of the didactical contract" among the two education levels concerning the teaching of geometry and further investigation is needed in order to gather information regarding the actual teaching of geometry in primary and secondary schools.

The power of the didactical contract of Natural Axiomatic Geometry

In the case of geometry teaching in the secondary school, the emphasis on learning theorems and continuous practice with close tasks demanding the application of theorems may result in the application of these theorems even in cases that this is not necessary. For example, as a consequence of the continuous practice of the Pythagoras' theorem and the didactical contract formed during teaching, students consider that they are expected to apply Pythagoras' theorem any time a right triangle is involved in a geometrical figure. As we have noted in the results section, attempting to solve a task which could be solved with the mere application of the property of equal sides in a square, almost one fifth of the 8th graders in the present study applied Pythagoras' theorem in the rectangular triangle they identified in the given geometrical figure. The power of the didactical contract in secondary school geometry concerning the application of theorems, leads students to mechanically apply the theorems, especially those that involve an algorithm, feeling safer to use an algorithm than a geometrical property.

Teaching implications and further research

Most of the difficulties that have been identified and discussed in the present study concerning primary and secondary school students' attempts to solve geometrical problems are centred around the issue of the difficulties raised during the transition from Natural Geometry paradigm (where the objects are real, material) to Natural Axiomatic Geometry paradigm (where the objects are conceptual). Subsequently, one of the main goals during the teaching of geometry should be to help students progressively pass from a geometry where objects and their properties are controlled by perception to a geometry where they are controlled by explicitation of properties. But, as Houdement and Kuzniak (2003) note, students and their teachers are not necessarily situated in the same geometrical paradigm, so this is a source of educational misunderstanding. Therefore, we consider essentially important that (prospective) primary and secondary school mathematics teachers are aware of the existence of the different geometrical paradigms (Houdement, 2007) and of the difficulties arising from the fact that plane geometrical figures on paper may be considered by the students in the teaching process during elementary school as if they were real objects (Berthelot & Salin, 1998). Further research is needed in order to prescribe and compare the way mathematics teachers in primary and secondary school approach geometry in their classrooms.

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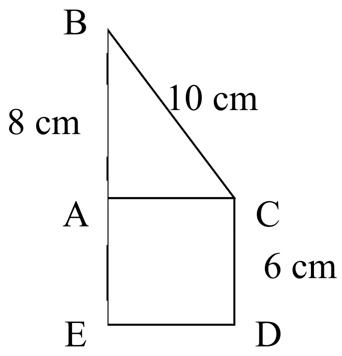
APPENDIX

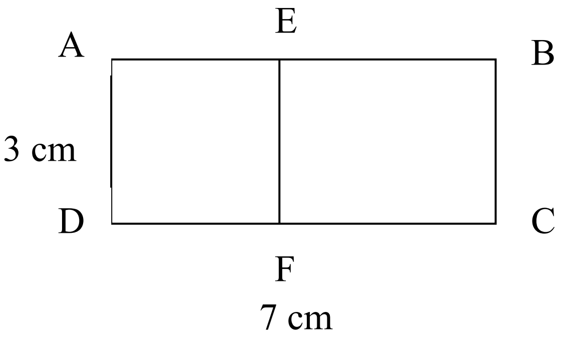
Item A

On the right triangle ABC, BC=10cm and AB=8cm. ACDE is a square (CD=6cm) . Find the length of segment AC.



Item C

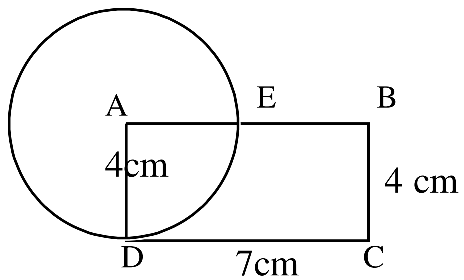
On the rectangle ABCD, DC=7cm and AD=3 cm. AEFD is a square. Find the length of segment EB.



Item B

On the figure sketched freehand here (the real lengths are written in cm), are represented a rectangle ABCD and a circle with center A, passing through D.

Find the length of segment EB.



STRENGTHENING STUDENTS' UNDERSTANDING OF 'PROOF' IN GEOMETRY IN LOWER SECONDARY SCHOOL

Susumu Kunimune, Taro Fujita & Keith Jones

Shizuoka University, Japan; University of Plymouth, UK; University of Southampton, UK

This paper reports findings that indicate that as many as 80% of lower secondary age students can continue to consider that experimental verifications are enough to demonstrate that geometrical statements are true - even while, at the same time, understanding that proof is required to demonstrate that geometrical statements are true. Further data show that attending more closely to the matter of the 'Generality of proof' can disturb students' beliefs about experimental verification and make deductive proof meaningful for them.

Key words: Geometrical reasoning, generality of proof, cognitive development, lower secondary school, curriculum design

INTRODUCTION

School geometry is commonly regarded as a key topic within which to teach mathematical argumentation and proof and to develop students' deductive reasoning and creative thinking. Yet while deductive reasoning and proof is central to making progress in mathematics, it remains the case that students at the lower secondary school level have great difficulty in constructing and understanding proof in geometry (Battista, 2007; Mariotti, 2007). Our work focuses on researching, and comparing, the teaching of geometry at the lower secondary school level in countries in the East and in the West, specifically China, Japan and the UK (see, for example, Ding, Fujita, & Jones, 2005; Ding & Jones, 2007; Jones, Fujita & Ding, 2004, 2005). In our research we are interested in students' cognitive needs in the learning of geometrical concepts and thinking, and in principles for classroom practice which would satisfy such needs of students.

In this paper we report selected findings from a series of research projects on the learning and teaching of geometrical proof carried out in Japan where formal proof is intensively taught in the lower secondary school grades (Grades 7-9). We address the issue of students' cognitive needs for conviction and verification and how these needs might be changed and developed through instructional activity. We first present how students in lower secondary schools perceive 'proof' in geometry in terms of the levels of understanding of geometrical proof. We do this by using data collected in 2005 from 418 Japanese students (206 from Grade 8, and 212 from Grade 9). We then offer some suggestions that we have developed from classroom-based research (undertaken since the 1980s) about how we might encourage students' geometrical thinking and understanding of deductive proof in geometry.

Given our data is from studies conducted in Japan, we begin with a short account of the teaching of proof in geometry in Japan.

THE TEACHING OF PROOF IN GEOMETRY IN JAPAN

The specification of the mathematics curriculum for Japan, the 'Course of Study', can be found in the *Mathematics Programme in Japan* (English edition published by the Japanese Society of Mathematics Education, 2000). It should be noted that no differentiation is required in the 'Course of Study', and mixed-attainment classes are common in Japan. 'Geometry' is one of the important topics (the other topics are 'Number and Algebra' and 'Quantitative Relations'). The curriculum states that, in geometry, students must be taught to "understand the significance and methodology of proof" (JSME, 2000, p. 24). In terms of the *Paradigm of Geometry* proposed by Houdement and Kuzniak (Houdement & Kuzniak, 2003), Japanese geometry teaching may be characterized as within the *Geometry II* paradigm (in that axioms are not necessarily explicit and are as close as possible to natural intuition of space as experienced by students in their normal lives).

In terms of Japanese curriculum materials (such as textbooks for Grade 8 and Grade 9 students) our analysis indicates a varying amount of emphasis on 'justifying and proving' (see, for example, Fujita and Jones, 2003; Fujita, Jones and Kunimune, 2008). While the curriculum requires that the principles of how to proceed with mathematical proof are explained in detail, including explanations of 'definitions' and 'mathematical proof', our research repeatedly shows that many students difficulties to understand proof in geometry (for example, Kunimune, 1987; 2000[1]).

In what follows we provide an analytical framework for students' understanding of proof in geometry and then report on our data from three from surveys carried out in 1987, 2000 and 2005.

ASPECTS OF STUDENTS' UNDERSTANDING OF PROOF IN GEOMETRY

In our research, as summarized in this paper, we capture students' understanding of proof in terms of two components: 'Generality of proof' and 'Construction of proof'. The first one these, 'Generality of proof in geometry', recognizes that, on the one hand, students have to understand the generality of proof in geometry, including the universality and generality of geometrical theorems (proved statements), the roles of figures, the difference between formal proof and experimental verification, and so on. The second of these two components, 'Construction of proof in geometry', recognizes that, on the other hand, students also have to learn how to 'construct' deductive arguments in geometry by knowing sufficient about definitions, assumptions, proofs, theorems, logical circularity, and so on.

Considering these two aspects, we work with the following levels of student understanding (we do not have space in this paper to relate these levels to the van Hiele model):

Level I: at this level, students consider experimental verifications are enough to demonstrate that geometrical statements are true. This level is sub-divided into two sub-levels: Level Ia: Do not achieve both 'Generality of proof' and 'Construction of proof', and Level Ib: Achieved 'Construction of proof' but not 'Generality of proof'

Level II: at this level, students understand that proof is required to demonstrate geometrical statements are true. This level is sub-divided into two sub-levels: Level IIa: Achieved 'Generality of proof', but not understand logical circularity, and Level IIb: Understood logical circularity

Level III: at this level, students can understand simple logical chains between theorems

We used the following questions to measure students' levels of understanding:

Q1 Read the following explanations by three students who demonstrate why the sum of inner angles of triangle is 180 degree.

Student A 'I measured each angle, and they are 50, 53 and 77. 50+53+77=180. Therefore, the sum is 180 degree.' Accept/Not accept

Student B 'I drew a triangle and cut each angle and put them together. They formed a straight line. Therefore, the sum is 180 degree.' Accept/Not accept

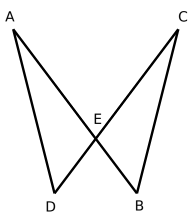
Student C Demonstration by using properties of parallel line (an acceptable proof) Accept/Not accept

Q2 In Figure Q2, prove AD = CB when ∠ A = ∠ C, and AE=CE.

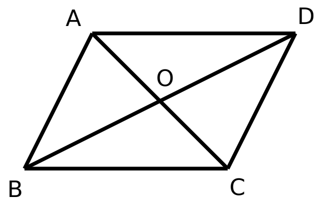
Q3The following argument carefully demonstrates that *the diagonals of a parallelogram intersect at their middle points* (see Figure Q3). 'In a parallelogram ABCD, let O be the intersection of its diagonals. In Δ ABO and Δ CDO, AB // DC. Therefore, ∠ BAO = ∠ DCO and ∠ ABO = ∠ CDO. Also, AB = CD. Therefore Δ ABO ≡ Δ CDO. Therefore, AO = CO and BO = DO, i.e. the diagonals of a parallelogram intersect at their middle points'

Now, why can we say a) AB // DC, b) AB = CD, and c) Δ ABO ≡ Δ CDO?

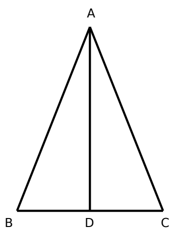
Q4 Do you accept the following argument which demonstrates that *in an isosceles triangle ABC, the base angles are equal*? (see Figure Q4). 'Draw an angle bisector AD from ∠ A. In Δ ABD and Δ ACD, AB = AC, ∠ BAD = ∠ CAD and ∠ B = ∠ C. Therefore, Δ ABD ≡ Δ ACD and hence ∠ B = ∠ C'. If you do not accept, then write down your reason.



Q2



Q3



Q4

In the above items, Question 1 (Q1) checks whether learners can understand difference between experimental verification and formal proof in geometry. Question 2 (Q2) checks whether learners can understand a simple proof. Q3 checks whether learners can identify assumptions, conclusions and so on in formal proof. Finally, Q4 checks whether learners can identify logical circularity within a formal proof (proof is invalid as '$\angle B = \angle C$' is used to prove '$\angle B = \angle C$'). To achieve Level II, students have to answer Q1 correctly. Students who perform well in Q2 and Q3 can be considered at least at Level Ib as they achieve good understanding in 'Construction of proof'. Figure 1 summarizes the criteria and levels.

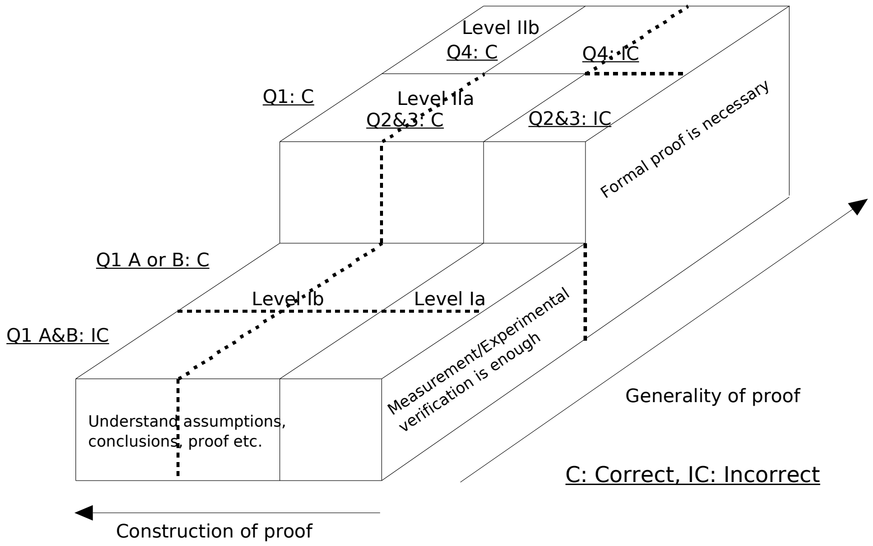


Figure 1: Criteria and levels of generality and proof construction

STUDENTS' UNDERSTANDING OF PROOF IN GEOMETRY

Student surveys were carried out in 1987, 2000 and 2005. One consistent result from these surveys is that over 60% students consider that experimental verification is enough to say it is true that the sum of the inner angles of triangle is 180 degree. Tables 1 and 2 show data collected in 2005 (with 206 students from Grade 8, and 212 students from Grade 9, collected from five different schools).

	Empirical argument using measures (Student A explanation)		**Empirical argument using tearing corners (Student B explanation)**		**Proof (Student C explanation)**	
	Accept	Not accept	Accept	Not accept	Accept	Not accept
Grade 8	62%	32%	70%	21%	74%	15%
Grade 9	36%	58%	52%	38%	80%	6%

Table 1: Results of Q1

The results in Table 1 indicate that, whereas students can accept (or understand) that a formal proof ('Student C' explanation) is a valid way of verification, many also consider experimental verification ('Student A' or 'Student B' explanation) as acceptable. There are, however, changes from Grade 8 to Grade 9, as, by the later grade, more students reject empirical arguments or demonstrations. The likely reason for this is that Grade 9 students have more experience with formal proof, whereas in Grade 8 the students are only just started studying proof (for more on this, see Fujita and Jones, 2003).

Turning now to students' understanding of 'Generality of proof' and 'Construction of proof', the results in Table 2 indicate the following:

- More than half of students can construct a simple proof (Q2).
- Students (in Q3) show relatively good performance for Q3a and Q3b, and these indicate that students have good understanding about deductive arguments of simple properties. Q3c is more difficult as students are required to have knowledge about the conditions of congruent triangles.
- The results of Q4 suggest that more than half of students cannot 'see' why the proof in Q4 is invalid; that is they cannot understand the logical circularity in this proof.

	Q2	Q3a	Q3b	Q3c	Q4
Grade 8	57%	82%	80%	53%	34%
Grade 9	63%	85%	81%	59%	49%

Table 2: Result of Q2-4

In summary, as shown in Table 3, some 90% of Grade 8 and 77% of Grade 9 students were found to be at level I. Data from surveys carried out in 1987 and 2000 show similar results (see Kunimune, 1987, 2000).

Level	**Ia**	**Ib**	**IIa or above**
Grade 8	33%	57%	9%
Grade 9	28%	49%	22%

Table 3: levels of understanding

The result from Grade 9 shows a sight improvement from Grade 8. Using a 2x2 cross-table in which the numbers of level Ia+Ib and IIa or above are considered, the chi-square value is 13.185 ($df=1$, $p<0.01$), and this indicates that the significant improvement can be observed between Grade 8 and Grade 9.

	Level Ia+Ib	**Level IIa or above**
Grade 8	185	19
Grade 9	163	47

Table 4: comparing Grade 8 and Grade 9

MOVING STUDENTS TO DEDUCTIVE THINKING

As evident in a recent review of research on proof and proving by Mariotti (2007, p181), the 'discrepancy' between experimental verifications and deductive reasoning is now a recognized problem. Japan is not an exception to this. Our findings given above indicate that Japanese Grade 8 and 9 students are achieving reasonably well in terms of 'Construction of proof', but not necessarily as well in terms of 'Generality of proof' in geometry. There is a gap between the two aspects. This means that students might be able to 'construct' a formal proof, yet they may not appreciate the significance of such formal proof in geometry. They may believe that a formal proof is a valid argument, while, at the same time, they also believe experimental verification is equally acceptable to 'ensure' universality and generality of geometrical theorems.

Our data for Grade 9 students can be considered as quite concerning, given 80% of students remain at level I in terms of their understanding of proof even after they have studied formal proof at Grade 8 using textbooks where 90% of relevant intended lessons can be devoted for 'justifying and proving' geometrical facts' (Fujita and Jones, 2003). However, we would like to stress that we are still encouraged by the result that 20% of Japanese students achieve relatively sound understanding of proof through everyday mathematics lessons.

Hence, in our research, we turn to the question of working with students on why formal proof is needed. Based on over 10 years of classroom-based research, Kunimune *et al* (2007) propose the following principles for lower secondary school geometry (Grades 7-9) designed to help students appreciate the *need* for formal proofs (in addition to the students being able to construct such proofs):

- Grade 7 lessons to start from problem solving situations such as 'consider how to draw diagonals of a cuboid', and so on; *this develops students' geometrical thinking and provides experiences of mathematical processes that are useful in studying deductive proofs in Grades 8 and 9*;
- Geometrical constructions to be taught in Grade 8; *this replaces the practice of teaching constructions in Grade 7, and then proving these same constructions in Grade 8, as such a gap between the teaching of constructions and their proofs has been found by classroom research to be unhelpful*;
- Grade 8 lessons to provide students with explicit opportunities to examine differences between experimental verifications and deductive proof; *this helps students to appreciate such differences*;
- Grade 8 lessons to start the teaching of the teaching of deductive geometry with a set of already learnt properties which are shared and discussed within the classroom, and used as a form of axioms (a similar idea to that of the 'germ theorems' of Bartolini Bussi, 1996); *this provides students with known starting points for their proofs*.

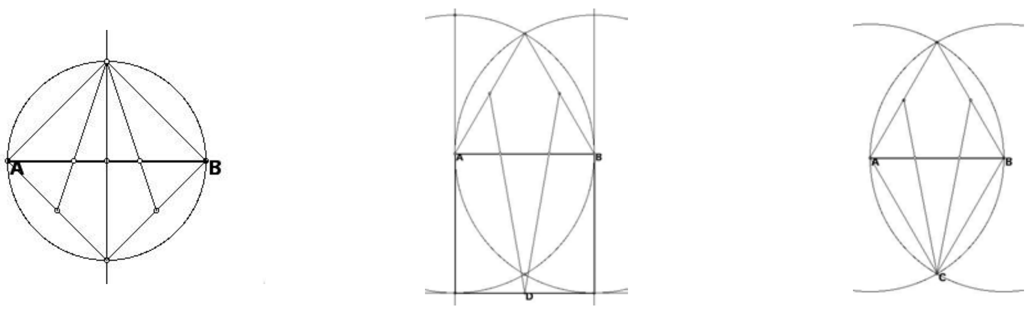
While we do not have space in this paper to provide data to support all these principles, in what follows we substantiate those related to differences between experimental verifications and deductive proof in geometry.

Constructions and proofs

In our experience (Shinba, Sonoda and Kunimune, 2004), while geometrical constructions (with ruler and compasses) can be taught in Grade 7, these constructions are often not proved until Grade 8 (after students have learnt how to prove simple geometrical statements). In a series of teaching experiments, we investigated the use of more complex geometrical constructions (and their proofs) in Grade 8. As an example, one of our lessons in Grade 8 started from the more challenging construction problem ‘Let us consider how we can trisect a given straight line AB’.

In our classroom studies, we observed that such lessons are more active for the students. The students could also experience some important processes which bridge between conjecturing and proving. Students could first investigate theorems/properties of geometrical figures through construction activities, and this led them to consider *why* the construction worked. By appropriate instructions by the teachers, the students then started proving the constructions. For example:

Student C: I thought that I could trisect AB when I constructed this (*No. 11* in Figure 2), but I think I found this is not true. So I prove that we cannot trisect the line AB. We just saw the construction No. 8 is true, so I use this approach in my proof. Now, I draw an equilateral triangle on AB (*No. 11’*), and by doing this, we can trisect the AB, and proof is similar to No. 8. Now, compare to this (*No. 11’*) to my construction, and C and D are not in the same place, as the height of the triangle ACB is shorter than the height of the square. We know we can trisect the AB by using this approach, and therefore, my method (*No. 11*) does not work.



No. 8 No. 11 No. 11’

Figure 2: Constructions proposed by students[2]

The data extract above shows that some students in this class start using an already proved statement (i.e. a theorem) to justify why the construction (No. 11 in Figure 2) does not work to trisect the line AB.

Making explicit the differences amongst various argumentations

In a series of lessons for 41 Grade 8 students, tasks were designed and implemented to disturb students' beliefs about experimental verification. In the lessons, students were asked, for example, to compare and discuss various ways of verifying the geometrical statement that the sum of the inner angles of triangles is 180 degrees (this relates to Q1 in the research questionnaire). The angle sum statement was chosen as way of trying to bridge the gap between empirical and deductive approaches, given that students often encounter the angle sum statement in primary schools and they study this again with deductive proof in lower secondary school. While we do not have space in this paper to provide the data from the study, we can provide a summary of ways which can be useful in encouraging students to develop an appreciation of why formal proof is necessary in geometry (for more details, see Kunimune, 1987; 2000).

- Students first exchange their ideas on various ways of verification; they comment on accuracy or generality of experimental verification; they discuss the advantages/disadvantages of experimental verifications.
- Students' comments such as 'A protractor is not always accurate ...', 'It takes time to measure angles, and we cannot see the reason why', 'The triangle is not general', and so on, often cause a state of disequilibrium in students (*viz* Piaget), and make students doubt the universality and generality of experimental verification.
- Students made various comment s on the argument based on 'cutting each angles and fitting them together' (Q1-b). For example, 'I think this is an excellent method as I cannot see any problems in this method', 'This is an easy method to check (whether the sum of inner angles of triangles is 180 degree), 'I think this is a good way, but because we use a piece of paper, I think it can be sometimes inaccurate', and so on.
- Advice from teachers is necessary to encourage students to reflect critically on different ways of verifications (*viz* establishment of 'social norm' in classrooms, Yackel and Cobb, 1996).

Kunimune (1987; 2000) found that, after such lessons, around 40% of students previously at Level Ib have moved to Level II (post-test I). They no longer accept experimental verification and start considering that deductive proof as the only acceptable argument in geometry. A later post-test (post-test II) carried out one month after the lessons found that about 60% of students are at Level IIa. Table 4 (below) summarises the result of the pre and post-tests with five types of cognitive changes observed among students in terms of the levels of understanding of proof in geometry.

An interesting observation is the type *d* in which three students show unexpected behaviour in terms of their cognitive development in that there was a regression from level IIa to Ib. A detailed reason for this is unknown, but, unlike the majority of students, it might be that their states of disequilibrium created rather a 'negative' effect for these students.

In summary, we conclude that the matter of the 'Generality of proof' could usefully be explicitly addressed in geometry lessons in lower secondary schools.

Type	Pre-test	Post-test I	Post-test II	N
a	Level II	Level II	Level II	2
b	Level I	Level II	Level II	13
c	Level I	Level I	Level II	9
d	Level I	Level II	Level I	3
e	Level I	Level I	Level I	14
Level II	2	18	24	

Table 4: Results from Pre- and Post tests

CONCLUDING COMMENTS

This paper outlines research findings from Japan suggesting that, in terms of 'Generality of proof' and 'Construction of proof', many students in lower secondary school remain at Level I where they hold the view that experimental verifications are enough to demonstrate that geometrical statements are true, even after intensive instruction in how to proceed with proofs in geometry. Classroom studies have tested ways of challenging such views about empirical ways of verification which indicate that it is necessary to establish classroom discussions to disturb students' beliefs about experimental verification and to make deductive proof meaningful for them.

NOTES

1. Some papers by Kunimune (1987; 2000) are written in Japanese; this paper, one of outcomes of our collaborative work over five years, contains his main ideas.
2. In No 8 AB is trisected by constructing a square whose diagonal is AB, and joining a vertex and midpoints; In No 11, an equilateral triangle and a square are constructed on AB; In No. 11', AB is trisected by constructing equilateral triangles on AB, and joining a vertex and midpoints.

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WRITTEN REPORT IN LEARNING GEOMETRY: EXPLANATION AND ARGUMENTATION

Sílvia Semana, Projecto AREA

Leonor Santos, DEFCUL, CIE, DIFMAT, Projecto AREA

In this article, we examine how the written report, within the context of assessment for learning, helps students in learning geometry and in developing their explanation and argumentation skills. We present the results of a qualitative case study involving Portuguese students of the 8th grade. This study suggests that using written reports improves those capabilities and, therefore, the comprehension of geometric concepts and processes. These benefits for learning are enhanced through the implementation of some assessment strategies, namely oral and written feedback.

Key-words: Geometric thinking, explanation, argumentation, assessment for learning, written reports.

INTRODUCTION

Explanation, argumentation and proof are mathematics activities that assume a main role in the teaching and learning of geometry, but present a lot of difficulties to students (Battista, 2007). The need to implement an assessment that contributes to students' learning is also widely recognized: an assessment that guides the students and helps them to improve their learning (Wiliam, 2007). As such, in this study, we attempted to understand how the written report, as a tool of assessment for learning, contributes to learning geometry and, in particular, reinforces the development of students' explanation and argumentation processes.

The present study follows a wider one that aimed at understanding the key role of the written report as an assessment tool supporting the learning of 8th grade students (aged thirteen) in mathematics. The larger study was developed during the academic year 2007/2008 under the scope of project AREA [1].

EXPLANATION, ARGUMENTATION AND PROOF IN TEACHING AND LEARNING GEOMETRY

All over the world and in Portugal, in particular, the mathematics curriculum recognizes geometry as a privileged field for the development of explanation, argumentation and proof (NCTM, 2000; DGIDC, 2007). Battista and Clements (1995) notice the need to shape the curriculum in order to develop students' explanation and argumentation skills and so that students use proof to justify powerful ideas. According to Polya (1957) mathematical proof should be taught because it helps in: (i) acquiring the notion of intuitive proof and logical reasoning; (ii) understanding a logical system; and (iii) keeping what is learnt in one's memory.

Many authors have addressed geometrical thought based on Van Hiele's model. This model proposes a sequential progression in learning geometry through five discrete and qualitatively different levels of geometrical thinking: visual, descriptive/analytic, abstract/relational, formal deduction and rigor. However, according to Freudenthal (1991), these are relative levels, not absolute ones. Nevertheless, "the levels can help to find and further develop appropriate tasks (…) and they are obviously helpful for explorative activities to come across new, maybe even innovative ideas" (Dorier *et al*., 2003, p. 2). This progression is determined by the teaching process, thus the teacher has a key role in setting appropriate tasks so that students may progress to higher levels of thought and walk towards proof. The learning of deductive proof in mathematics is complex and its progress is neither linear nor free of difficulties (Küchemann & Hoyle, 2002, 2003). As regards explanation, we may consider several modes, including non-explanations (where, for example, students refer to the teacher's authority), explaining how, explaining to someone else (spontaneously) and explaining to oneself (in response to a question) (Reid, 1999). Argumentation is view as an intentional explication of the reasonings used during the development of a mathematical task (Forman *et al*., 1998).

ASSESSMENT FOR LEARNING

Current mathematics curriculum documents advocate an assessment whose main purpose is to support students' learning, and whose forms constitute, at the same time, learning situations (DGIDC, 2007; NCTM, 1995, 2000). "Assessment in education must, first and foremost, serve the purpose of supporting learning" (Black & Wiliam, 2006, p. 9). In this study, assessment for learning is seen as "all the intent that, acting on the mechanisms of learning, directly contributes to the progression and/or redirection of learning" (Santos, 2002, p. 77). Several studies show that the focus on assessment for learning, as opposed to an assessment of learning, may produce substantial improvement in the performance of students (Black & William, 1998).

In order to develop their own knowledge about thinking mathematically, students need to develop a conscious, reflective practice, which encompasses the processes of self-assessment. According to Hadji (1997), self-assessment is an activity of reflected self-control over actions and behaviours on behalf of the individual who is learning. Santos (2002) stresses that self-assessment implies that one becomes aware of the different moments and aspects of his/her cognitive activity, therefore it is a meta-cognitive process. A non-conscious self-control action is a tacit, spontaneous activity that is natural in the activity of any individual (Nunziati, 1990), and in this sense all human beings self-assess themselves. Meta-cognition goes beyond non-conscious self-control, for it is a conscious and reflective action (Nunziati, 1990).

Some assessment strategies can be adopted to promote learning, including: a positive approach of the error; oral questioning of students; feedback; negotiation of assessment criteria; and the use of alternative and diversified assessment instruments (Black *et al*., 2003; Santos, 2002). In particular, the written report is a privileged instrument to monitor students' learning. Students' work on written reports has advantages in terms of developing their explanation and argumentation skills, which are two intrinsic requests of this instrument; furthermore, written reports may help students to reflect upon their work, because time and space are given (Mason, Burton & Stacey, 1982). "Intensive approach to argumentative skills, relevant for mathematical argumentation, seems to be possible through an interactive management of students' approach to writing" (Douek & Pichat, 2003). The description of thinking processes, with the identification of the strategies used to solve a given task, including the difficulties that were encountered and the mistakes that were made, allows students to rethink their learning process. However, it is desirable that a report be done in "two stages" to allow for an effective opportunity for learning. This means that a first version of the report is subject to the teacher's feedback and then the student develops a new version, a second one, taking into account the feedback received (Pinto & Santos, 2006).

METHODOLOGY

This study was based on an interpretative paradigm and on a qualitative approach. We chose the case study for the design research, given the nature of the problem to study and the desired final product (Yin, 2002).

The research involved an 8th grade class, with 24 students. We selected four of these students based on different mathematical performances, and taking into account their mathematics communication skills. These students were Maria, Rute, Duarte, and Telmo, and they constituted a working group in the classroom.

Data were collected through lesson observation, namely, the lesson dedicated to the discussion of the guidelines for preparing the report and of the assessment criteria, and the lessons dedicated to carrying out tasks as well as the first and second versions of the reports. Three individual interviews to each of the four students were made, the first one at the beginning of the school year and the others after the establishment of the second version of each report. Two tasks led to the development of two written reports, each one with two versions.

The data were subjected to several levels of analysis that took place periodically (Miles & Huberman, 1994), based on categories defined a posteriori that arose from the data gathered, keeping in mind the focus of the study and the theoretical framework.

PEDAGOGICAL CONTEXT

Since the writing of a report was a novelty for the students, they were given a set of guidelines for writing the report and the assessment criteria. These two documents were discussed with the students. According to the guidelines, the organization of the report should include three parts: introduction, development, and conclusion. Both first two parts, and the tasks that originated the report, should be produced within the group. The last part should be held individually and it included students' self-assessment. The reports were produced in two "stages", the students benefiting from the teacher's comments to the first stage in order to improve the second one. Students were not required to do any proof, but were asked to provide explanations for their thinking (Küchemann & Hoyle, 2003).

The first task proposed an investigation of possible generalizations of the Pythagorean theorem. Students were asked to remember and to reflect upon the relationship between the areas of the squares constructed on the sides of a right triangle, and to investigate what happens if they construct other geometric figures on the sides of a right triangle. The second task was a problem that involves the application of the Pythagorean theorem in space. Students were asked to construct a cone based on one of the three equal sectors of a circle, with a radius of six centimetres, and to determine the height of the constructed cone. They were also encouraged to explain how they could determine the height of a cone obtained from a circle with a radius r. These tasks were chosen based on the assumption that presenting students with unfamiliar questions can provide a rich context for classroom discussion which helps students in developing mathematical arguments (Küchemann & Hoyle, 2003).

The first report

In the first task, students reflect on the meaning and implications of the Pythagorean theorem and review some geometric concepts and procedures (such as what an equilateral triangle is and how it can be constructed with ruler and compass). Due to the nature of the task, the group is still required to formulate and test conjectures, and to argue in favour of their ideas, thus appealing to students' mathematical reasoning skills. In particular, when writing the report, the students, in group, explain how they exploited the first situation proposed in the task, concerning equilateral triangles built on the sides of a right triangle.

In the first version of their report, students described how they had built the equilateral triangles and stated how they had determined the areas of those triangles:

> We started by making a right triangle, with the help of a compass we drew around it (at the endpoints of the right triangle) three equilateral triangles, because we couldn't obtain equilateral triangles nor a good graphic design by using rules. We determined the area of the triangles.

The justification for the use of compass comes in the wake of some oral feedback provided during the preparation of the report. This feedback may have helped the students to explain their options:

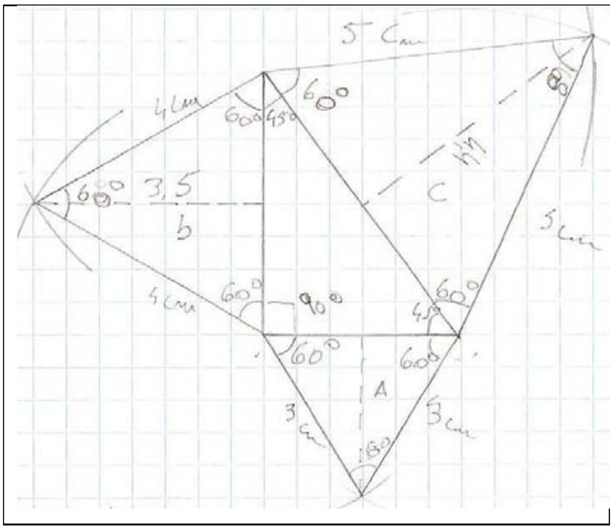
Rute: We did it like this: with the help of the compass, we made around it three equilateral triangles. Then we can put… ah…

Teacher: Why did you use the compass?

Rute: Because we couldn't complete the task with the ruler only.

Teacher: So, couldn't you draw a triangle with the ruler only?

Rute: Yes, but in order to be an equilateral triangle, it had to have all equal sides.



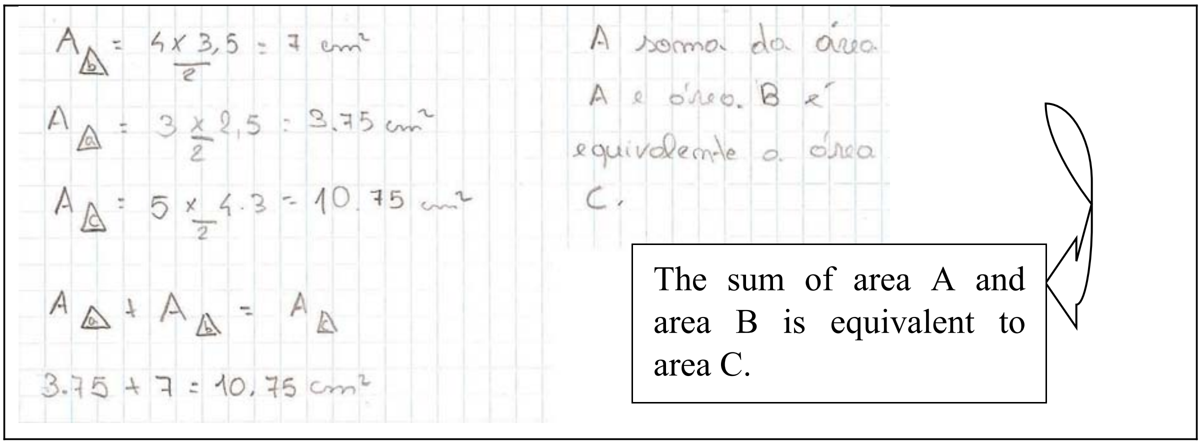
In an attached document to their report, the group presented the construction of equilateral triangles, as well as the values of the basis and the height considered in each one. It also presented the calculations that were made to determine the corresponding areas.

However, in any part of the report, did the students explain how they had found the values of the bases and heights, nor what conclusions they obtained from the areas determined. Two different comments were provided to the first version of the report. On the one hand, the teacher praised students for their use of a compass and the reasons for their choice: "You did an excellent option. It's a good way to answer a problem that you had to overcome." In this way, the teacher identified positive aspects of the report, so that knowledge could be consciously recognized by students and their self confidence could be promoted (Santos, 2003). On the other hand, the teacher questioned students about the conclusions they had drawn from the areas obtained: "And what did you find?". Furthermore, the teacher still posed some questions written near the construction of the triangles, which sought to guide the work of students in order to include the missing information in the report: "How did you come to these figures? Which relationship may you establish?"

While working on the second version of their report, the students kept the description that had been praised and tried to answer the questions. They explained in more detail how they had proceeded, namely in finding the values of the basis and height of the triangles, in determining the corresponding areas in each equilateral triangle, and in making explicit the conclusions they had obtained for the first situation:

> We determined the area of the triangles. We know that in order to determine the area of a triangle: $\frac{basis \times height}{2}$, we measured the height and the basis, we multiplied and then we divided by 2 (and likewise for the three triangles). We concluded that the sum of area A and area B is equal to area C.

In the final version, the students determined and identified the value of the area of each one of the considered triangles and explained the relationship found among the areas of the equilateral triangles constructed on the sides of the right triangle. This work was based on the figure of the first version:



$A_{\triangle} = \frac{4 \times 3,5}{2} = 7\ cm^2$

$A_{\triangle} = \frac{3 \times 2,5}{2} = 3.75\ cm^2$

$A_{\triangle} = \frac{5 \times 4.3}{2} = 10.75\ cm^2$

$A_{\triangle} + A_{\triangle} = A_{\triangle}$

$3.75 + 7 = 10,75\ cm^2$

A soma da área A e área B é equivalente a área C.

The sum of area A and area B is equivalent to area C.

Students still added a comment. They identified the negative aspects of the first version and they improved them in the second stage: "[In the first stage] we didn't present the value for the areas, we messed up the computations, and we did not present the conclusions." The students identified and corrected their own mistakes.

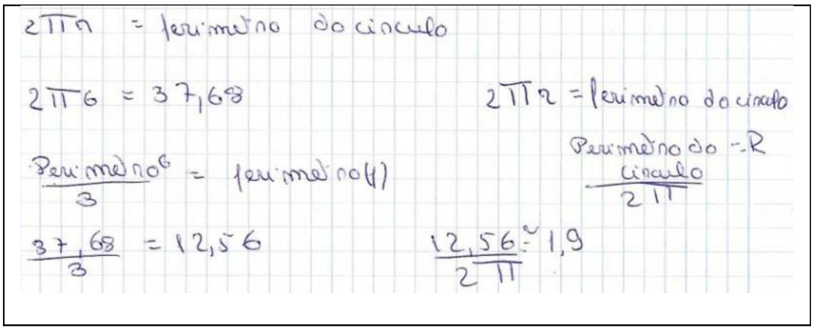
The second report

In the second task, the students review and apply the Pythagorean theorem as well as some mathematical concepts and procedures (such as, the height of a cone or the perimeter of a circle given its radius). Due to the nature of the task, it calls, mostly, for problem-solving and mathematical reasoning skills.

In the report, the students explained how they had built the cones and sought reasons for their actions. In particular, they explain how to determine the angle of each of the three circular sectors:

> We started by reading the task and answering to what had been requested. We drew a circle of radius 6 cm. To divide the angle into three equal parts, we know that the angle measures 360º: (so $\frac{360^{o}}{3} = 120^{o}$). With the help of a protractor, we measured, on the radius, 120º three times and joined the points and we got 3 equal parts. Then, we cut the three parts, and with the help of some tape, we constructed three cones.

Then, the students described the strategy implemented to determine the height of the cones. Before moving to the resolution itself, they made a brief description of how the group had addressed the issue, referring various ideas discussed and some difficulties encountered, which they sought to overcome with the help of the teacher. Then they determined the radius of the basis of the cone, giving the necessary calculations (determining the perimeter of the original circle, the perimeter of the basis of the cone and, finally, the radius of the basis of the cone).



$2\pi r$ = perímetro do círculo

$2\pi 6 = 37,68$

$2\pi r$ = perímetro do círculo

$\frac{\text{Perímetro do círculo}}{2\pi} = R$

$\frac{\text{Perímetro}}{3} = \text{perímetro}(1)$

$\frac{37,68}{3} = 12,56$

$\frac{12,56}{2\pi} \simeq 1,9$

However they did not explain the calculations nor did they give reasons for those calculations; they did not distinguish the two circles involved (the original one and the basis of the cone), nor did they present units of measurement. Written feedback was provided with the intention of alerting students to these aspects: "Why did you do these calculations? You refer the perimeter of the circle several times. Maybe it would be better to distinguish which circle you are talking about in each situation. Attention to the lack of measurement units". The importance of students' explanation and justification of their calculations was further strengthened through oral feedback:

Teacher: "(...) you must try to explain the calculations you presented better and why you have done them". You presented these calculations, didn't you? For what? When? How?

Rute: The teacher wants to know everything!

Teacher: I want to know everything, no… Imagine that I'm teaching a lesson and I write something on the blackboard, and then you ask me "teacher, what is that?" and I say "You want to know everything!", right?

Rute: Teacher, but, here, we already know that this is the perimeter...

Teacher: You know, but you must write what you mean. I am not going to take Rute home to explain it to me, right?

It was also necessary to complement the written feedback with new clues, so that the students could distinguish the different circles considerered in the resolution of the problem:

Rute: Teacher, how do we distinguish the circles?

Teacher: Which circles did you work with?

Rute: With the one with radius six.

Teacher: Yes. And didn't you work with any other circle?

Rute: With the basis.

Teacher: The basis?

Rute: Yes, of the cone.

Teacher: So, in the report, you only have to say which one you are referring to when you explain what you did.

The students took into account the feedback received, both oral and written. In the final version of the report, besides adding the measurement units, they described how they had proceeded to determine the radius of the basis of the cone. They clarified the context, they explained the purpose of the calculations they had presented, and they also identified the circle referred in each case:

> First we found the perimeter of the circle of the problem. Then we divided the perimeter of the circle of the problem into three equal parts, and we got the perimeter of

> the basis of a cone. Knowing that to find the perimeter of the circle is $2\pi\ r$, to find the radius is the other way around: $P \div 2\pi = \mathrm{r}$. And then, we obtained 1,9 cm.

In the first version of the report, students had already tried to describe in detail the right triangle used to determine the height of the cone and they explained how they had determined the length of the hypotenuse (which they refer to as diagonal) of that triangle:

> If we draw the height of the cone, it will coincide with the radius forming an angle of 90º. If, at the endpoints of the lines, we draw a line segment, it will form a right triangle and, for our own luck, it was the diagonal, which we knew about.
>
> 
>
>
> We know that the diagonal measures 6 cm because the diagonal is the radius of the circle when we open the cone, and, as the radius of the circle is 6 cm, we got to know the diagonal.

Finally, the students presented the necessary calculations to determine the height of the cone, but they did not mention how they had concluded that "height of the cone^2 = diagonal^2 - radius^2". They were reminded of this fact through written feedback: "How do you achieve this equality?" In the final version of the report, the students considered the feedback received and stated that they had used the Pythagorean theorem to obtain the height of the cone.

DISCUSSION OF RESULTS

In this study, students were asked to describe and explain the strategies used in the implementation of two tasks and to submit the results, duly substantiated, under the form of written reports. Students, working in a group, were given constructive comments on the first version of their reports so that they could improve their work and develop a second version. In many cases, in the first version of the reports, students gave procedural explanations instead of providing a mathematical justification (Hoyle & Küchemann, 2003). In other words, they presented how they had done their work, but not why. For example, in the first version of the report regarding the first task, students described how they had built the equilateral triangle, but they did not mention the characteristics of this figure. In the second version of the report, students presented mathematical arguments for the choices made and for the results found in performing the tasks. They also used symbolic language of mathematics when necessary (it happened, for example, when they obtained the area of equilateral triangles in the first task or when they obtained the height of the cone in the second task). However, in both cases, they seemed to be, mainly, at the descriptive/analytic level of Van Hiele's geometrical thinking model.

Feedback, both oral and written, allowed students to identify aspects to improve in the reports and provided clues about what students could do to develop their first productions. Indeed, feedback seems to have enabled students to produce a better report in the second version, especially regarding explanation and justification of the strategies adopted (it should be noted, for example, the explanation given, in the final version, to the operation performed in the first phase to obtain the radius of the basis of the cone, starting from its perimeter). In addition, the feedback did not contain any information about errors; it only included guiding questions and comments (Black *et al*., 2003; Santos, 2003). This led students to identify mistakes and to correct them (as is evident in the first task, in which the students relate what they had done wrong in the first version). Thus, feedback also promoted the development of students' reflection and self-assessment skills (Nunziati, 1990).

The need for students to explain and justify, in written form, the mathematical procedures and results involved in performing mathematically rich tasks caused a high level of demand and consequently of learning. These situations, which involve knowledge that students possibly know, but which they need to explain and justify, have a strong didactic purpose (Küchemann & Hoyle, 2003). The identified benefits associated with the written reports seem to be enhanced by investing on a type of report in "two stages", in which oral and written feedback gain prominence.

NOTES

1. The project AREA (Monitoring Assessment in Teaching and Learning) is a research project funded by the Foundation for Science and Technology (PTDC/CED/64970/2006). The main objectives of the project are to develop, implement and study practices of assessment that contribute for learning. Further information can be found in http://area.fc.ul.

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MULTIPLE SOLUTIONS FOR A PROBLEM: A TOOL FOR EVALUATION OF MATHEMATICAL THINKING IN GEOMETRY

Anat Levav-Waynberg & Roza Leikin

University of Haifa - Israel

Based on the presumption that solving mathematical problems in different ways may serve as a double role tool - didactical and diagnostic, this paper describes a tool for the evaluation of the performance on multiple solution tasks (MST) in geometry. The tool is designed to enable the evaluation of subject's geometry knowledge and creativity as reflected from his solutions for a problem. The example provided for such evaluation is taken from an ongoing large-scale research aimed to examine the effectiveness of MSTs as a didactical tool. Geometry is a gold mine for MSTs and therefore an ideal focus for the present research, but the suggested tool could be used for different mathematical fields and different diagnostic purposes as well.

Introduction

The study described in this paper is a part of ongoing large-scale research (Anat Levav-Waynberg; in progress). The study is based on the position that solving mathematical problems in different ways is a tool for constructing mathematical connections, on the one hand (Polya, 1973, 1981; Schoenfeld, 1988; NCTM, 2000) and on the other hand it may serve as a diagnostic tool for evaluation of such knowledge (Krutetskii, 1976). In the larger study we attempt to examine how employment of Multiple-solution tasks (MSTs) in school practice develops students' knowledge of geometry and their creativity in the field. In this paper we present the way in which students' knowledge and creativity are evaluated.

Definition: MSTs are tasks that contain an explicit requirement for solving a problem in multiple ways. Based on Leikin & Levav-Waynberg (2007), the difference between the solutions may be reflected in using: (a) Different representations of a mathematical concept; (b) Different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic; or (c) Different mathematics tools and theorems from different branches of mathematics.

Note that in the case of MSTs in geometry we consider different auxiliary constructions as a difference of type (b).

Solution spaces

Leikin (2007) suggested the notion of "solution spaces" in order to examine mathematical creativity when solving problems with multiple solution approaches as follows: *Expert solution space* is the collection of solutions for a problem known to the researcher or an expert mathematician at a certain time. This space may expand as new solutions to a problem may be produced. There are two types of sub-sets of expert solution spaces: The first is *individual solution spaces* which are of two kinds. The distinction is related to an individual's ability to find solutions independently. *Available solution space* includes solutions that the individual may present on the spot or after some attempt without help from others. These solutions are triggered by a problem and may be performed by a solver independently. *Potential solution space* include solutions that solver produce with the help of others. The solutions correspond to the personal zone of proximal development (ZPD) (Vygotsky, 1978). The second subset of an expert space is *a collective solution space* characterizes solutions produced by a group of individuals.

In the present study solution spaces are used as a tool for exploring the students' mathematical knowledge and creativity. By comparing the individual solution spaces with the collective and expert solution spaces we evaluate the students' mathematical knowledge and creativity.

MST and mathematics understanding

The present study stems from the theoretical assumption that mathematical connections, including connections between different mathematical concepts, their properties, and representations form an essential part of mathematical understanding (e.g., Skemp, 1987; Hiebert & Carpenter, 1992; Sierpinska, 1994). Skemp (1987) described understanding as the connection and assimilation of new knowledge into a known suitable schema. Hiebert & Carpenter (1992) expanded this idea by describing mathematical understanding as "networks" of mathematical concepts, their properties, and their representations. Without connections, one must rely on his memory and remember many isolated concepts and procedures. Connecting mathematical ideas means linking new ideas to related ones and solving challenging mathematical tasks by seeking familiar concepts and procedures that may help in new situations. Showing that mathematical understanding is related to connectedness plays a double role: it strengthens the importance of MSTs as a tool for mathematics education and it justifies measuring mathematics understanding by means of observing the subjects' mathematical connections reflected from one performance on MSTs..

Why geometry

The fact that *proving* is a major component of geometry activity makes work in this field similar to that of mathematicians. The essence of mathematics is to make abstract arguments about general objects and to verify these arguments by proofs (Herbst & Brach, 2006; Schoenfeld, 1994).

If proving is the main activity in geometry, *deductive reasoning* is its main source. Mathematics educators claim that the deductive approach to mathematics deserves a prominent place in the curriculum as a dominant method for verification and validation of mathematical arguments, and because of its contribution to the development of logical reasoning and mathematics understanding (Hanna, 1996; Herbst & Brach, 2006). In addition to these attributes of geometry, which make it a meaningful subject for research in mathematics education, geometry is a gold mine for MSTs and therefore an ideal focus for the present research.

Assessment of creativity by using MST

Mathematical creativity is the ability to solve problems and/or to develop thinking in structures taking account of the peculiar logico-deductive nature of the discipline, and of the fitness of the generated concepts to integrate into the core of what is important in mathematics (Ervynck ,1991, p.47)

Ervynck (1991) describes creativity in mathematics as a meta-process, external to the theory of mathematics, leading to the creation of new mathematics. He maintains that the appearance of creativity in mathematics depends on the presence of some preliminary conditions. Learners need to have basic knowledge of mathematical tools and rules and should be able to relate previously unrelated concepts to generate a new product. The integration of existing knowledge with mathematical intuition, imagination, and inspiration, resulting in a mathematically accepted solution, is a creative act.

Krutetskii (1976), Ervynck (1991), and Silver (1997) connected the concept of creativity in mathematics with MSTs. Krutetskii (1976) used MSTs as a diagnostic tool for the assessment of creativity as part of the evaluation of mathematical ability. Dreyfus & Eisenberg (1986) linked the aesthetic aspects of mathematics (e.g., elegance of a proof/ a solution) to creativity. They claim that being familiar with the possibility of solving problems in different ways and with their assessment could serve as a drive for creativity. In sum, MSTs can serve as a medium for encouraging creativity on one hand and as a diagnostic tool for evaluating creativity on the other.

According to the Torrance Tests of Creative Thinking (TTCT) (Torrance, 1974), there are three assessable key components of creativity: fluency, flexibility, and originality. Leikin & Lev (2007) employed these components for detecting differences in mathematical creativity between gifted, proficient and regular students in order to explain how MSTs allow analysing students' mathematical creativity, and thus serve as an effective tool for identification of mathematical creativity.

Fluency refers to the number of ideas generated in response to a prompt, *flexibility* refers to the ability to shift from one approach to another, and *originality* is the rareness of the responses.

In order to assess mathematical thinking in the Hiebert & Carpenter (1992) and Skemp (1987) sense, while evaluating problem solving performance of the participants on MSTs, we added the criterion of connectedness of mathematical knowledge which is reflected in the overall number of concepts/theorems used in multiple solutions of a MST.

In this paper we outline the use of MSTs as a research tool for evaluation of mathematical knowledge and creativity in geometry.

Method

Following MST instructional approach, three 60 minutes tests were given to 3 groups of 10th grade, high-level students during geometry course (total number of 52 students). The first test was admitted in the beginning, the second in the middle and the third in the end of the course. Each test included 2 problems on which students were asked to give as many solutions as they can.

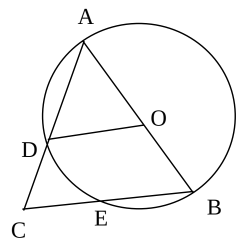
Example of the task

The following is one of the MSTs used for the tests

TASK:

AB is a diameter on circle with center O. D and E are points on circle O so that DO||EB .
C is the intersection point of AD and BE (see figure).

Prove ***in as many ways as you can*** that CB=AB



Examples of the solutions

Solution 1:

$DO = \frac{1}{2}AB$ (Equal radiuses in a circle) $\Rightarrow$ DO is a midline in triangle ABC (parallel to BC and bisecting AB) $\Rightarrow$ $DO = \frac{1}{2}AB = \frac{1}{2}BC \Rightarrow$ AB=BC

Solution 2:

DO=AO (Equal radiuses in a circle) $\Rightarrow$ $\angle AOD = \angle ABC$ (Equal corresponding angles within parallel lines) $\Rightarrow$ $\angle A = \angle A$ (Shared angle) $\Rightarrow$ $\Delta AOD \sim \Delta ABC$ (2 equal angles) $\Rightarrow$
AB=BC (a triangle similar to an isosceles triangle is also isosceles)

Solution 3:

DO=AO (Equal radiuses in a circle) $\Rightarrow$ $\angle ADO = \angle A$ (Base angles in an isosceles triangle)

$\angle ADO = \angle ACB$ (Equal corresponding angles within parallel lines), $\angle ACB = \angle A \Rightarrow$

AB=BC (a triangle with 2 equal angles in isosceles)

Solution 4:

Auxiliary construction: continue DO till point F so that DF is a diameter. Draw the line FB (as shown in the figure)

DO=AO (Equal radiuses in a circle) $\Rightarrow$ $\angle ADO = \angle A$ (Base angles in an isosceles triangle)

$\angle F = \angle A$ (Inscribed angles that subtend the same arc) $\Rightarrow$

$\angle F = \angle ADO \Rightarrow$ CD|| BF (equal alternate angles)

DFBC is a parallelogram (2 pairs of parallel sides) $\Rightarrow$

DF=CB (opposite sides of a parallelogram), DF=AB (diameters) $\Rightarrow$ AB=BC

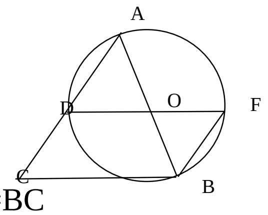


Figure 1: Example of MST

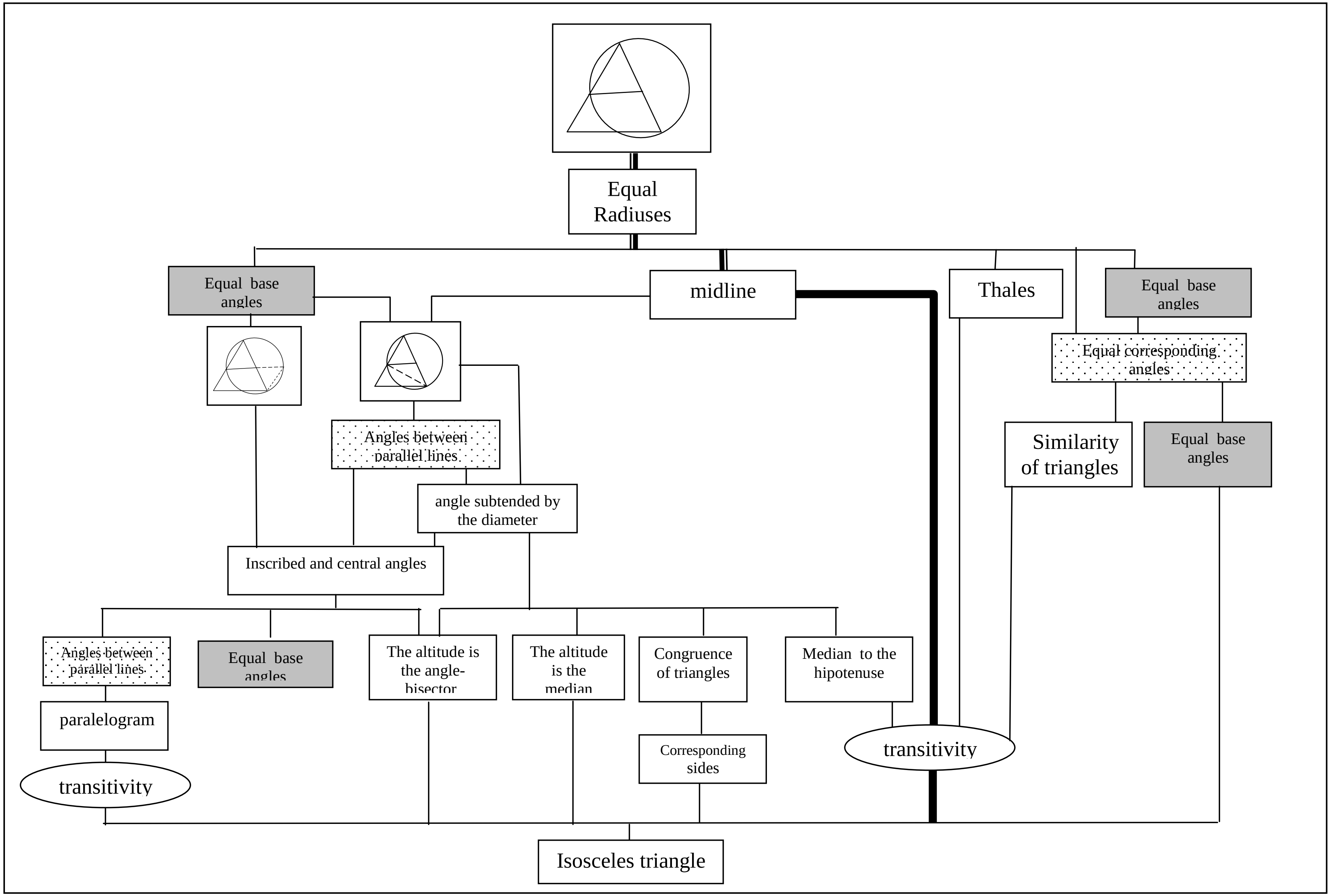


Figure 2: The map of an expert solution space for the task (see Figure 1)

Figure 1 presents an example of a task used in this study. Figure 2 depicts a map of the expert solution space for this task. The map outlines concepts and properties used in all the solutions as well as the order of their use in each particular solution (for additional maps of MSTs see Leikin, Levav-Waynberg, Gurevich and Mednikov, 2006).

The bold path in the map (Figure 2) represents Solution 1 of the task (see Figure 1).

Data analysis

<table>
<tr><th rowspan="2"></th><th rowspan="2">Correctness</th><th rowspan="2">Connectedness</th><th colspan="3">Creativity</th></tr>
<tr><th>Fluency</th><th>Flexibility
groups of solutions should be defined</th><th>Originality
According to (P) frequency (conventionality) of a strategy used</th></tr>
<tr><td>Scores per solution</td><td>1-100</td><td>--</td><td>1</td><td>Flx=10 for the first solution
Flxi=10 solutions from different groups of strategies
Flx_i=1 solutions from the same group – meaningfully different subgroups
Flxi=0.1 solutions from the same group-similar subgroups</td><td>Or_i=10 P<15%
Or_i=1 15%<P<40%
Or_i=0.1 P>40%</td></tr>
<tr><td rowspan="2">Final score for a solution group (strategy used by a student including its repetitions)</td><td>--</td><td>$\frac{t}{T}*100$</td><td></td><td>$Flx = \sum Flx_i$</td><td>$Or = \sum Or_i$</td></tr>
<tr><td></td><td></td><td colspan="3">$Flx_i \times Or_i$</td></tr>
<tr><td>Score per individual solution space for a problem (SPI)</td><td>Correctness= Max score among the solution for a problem</td><td>$Con = \frac{t}{T}*100$</td><td colspan="3">$n \times \sum (Fl_i \times Or_i)$</td></tr>
<tr><td colspan="4">n: number of solutions in the individual solution space
N: number of the students in a group
T: number of concepts and their properties used in the expert solution space
t: number of concepts and their properties used in the individual solution space</td><td colspan="2">m_i: the number of students who used the strategy i
$P = \frac{m_i}{N} 100\%$</td></tr>
</table>

Figure 3: Scoring scheme for the evaluation of problem-solving performance on a particular MST based on Leikin (forthcoming)

The analysis of data focuses on the student's individual solution spaces for each particular problem. The spaces are analyzed with respect to (a) Correctness; (b) Connectedness; (c) Creativity including fluency, flexibility, and originality.

The maximal correctness score for a solution is 100. It is scored according to the preciseness of the solution. When solution is imprecise but lead to a correct conclusion we consider it as *appropriate* (cf. Zazkis & Leikin, 2008). The highest correctness score in an individual solution space serves as the individual's total correctness score on the task. This way a student who presented only 1 correct solution (scored 100) does not get a higher correctness score than a student with more solutions but not all correct. Connectedness of knowledge associated with the task is determined by the total number of concepts and theorems in the individual solution space. Figure 3 depicts scoring scheme for the evaluation of problem-solving performance from the point of view of correctness, connectedness and creativity. The scoring of creativity of a solutions space is borrowed from Leikin (forthcoming). In order to use this scheme the expert solution space for the specific MST has to be divided into groups of solutions according to the amount of variation between them so that similar solutions are classified to the same group. The number of *all the appropriate solutions* in one's individual solution space indicates one's fluency while flexibility is measured by the *differences among acceptable solutions* in one's individual solution space. Originality of students' solution is measured by the rareness of the solution group in the mathematics class to which the student belongs. In this way a minor variation in a solution does not make it original since two solutions with minor differences belong to the same solution group.

Note that evaluation of creativity is independent of the evaluation of correctness and connectedness. In order to systematize the analysis and scoring of creativity and connectedness of one's mathematical knowledge we use the map of an expert solution space constructed for each problem (see Figure 2).

Results – example

In the space constrains of this paper we shortly exemplify evaluation of the problem-solving performance of three 10^{th} graders – Ben, Beth and Jo -- from a particular mathematics class. The analysis provided is for their performance on Task in Figure 1. Their solutions are also presented in this figure. We present these students' results because they demonstrate differences in fluency, flexibility and originality. Solutions 1, 2 and 3 are classified as part of the same solutions group whereas solution 4 which uses a special auxiliary construction is classified as part of a different group.

Ben performed solutions 1, 3 and 4, Beth produced solutions 1, 2 and 3, and Jo succeeded to solve the problem in two ways: solutions 1 and 3 (Figure 1). Figure 4 demonstrates connectedness and creativity scores these students got on the Task when the scoring scheme was applied (Figure 3). Their correctness score for all the solutions they presented was 100.

We observed the following properties of the individual solution spaces for Ben and Beth: they were of the same sizes; they included the same number of concepts and theorems and contained two common solutions (solutions 1 and 3). However Ben's creativity score was much higher then Beth's one as a result of the originality of Solution 4 that was performed only by Ben, and his higher flexibility scores.

Beth and Jo differed mainly in their fluency: Beth gave 3 solutions and Jo only 2. Since their solutions had similar flexibility and originality scores their creativity scores are proportional to their fluency scores.

		Solution Type (in order of presentation in the test)	**group**	**Connectedness**	**Creativity**			
					Fluency	Flexibility	Originality	creativity
Ben	Scores per solution	1	1			10	0.1	1
		3	1			1	0.1	0.1
		4	3			10	10	100
	Final			50	3			303.3
Beth	Scores per solution	2	1			10	0.1	1
		3	1			1	0.1	0.1
		1	1			1	0.1	0.1
	Final			50	3			3.6
Jo	Scores per solution	3	1			10	0.1	1
		1	1			1	0.1	0.1
	Final			30	2			2.2

Figure 4: Evaluation of the solutions on the task for three students

Concluding remarks

MSTs are presented in this paper as a research tool for the analysis of students' mathematical knowledge and creativity. The tasks are further used in the ongoing study in order to examine their effectiveness as a didactical tool. The larger study will perform a comparative analysis of students' knowledge and creativity along employment of MST in geometry classroom on the regular basis. The scoring scheme presented herein can be considered as an upgrading of the scoring scheme suggested by Leikin and Lev (2007). Correspondingly we suggest that the scoring scheme presented herein can be used for examination of individual differences in students' mathematical creativity and students' mathematical knowledge in different fields. We are also interested in employment of this tool for the analysis of the effectiveness of different types of mathematical classes in the development of students' mathematical knowledge and creativity.

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THE DRAG-MODE IN THREE DIMENSIONAL DYNAMIC GEOMETRY ENVIRONMENTS – TWO STUDIES

Mathias Hattermann

University of Giessen, Germany

Dynamic Geometry Environments (DGEs) in 2D are one of the well researched topics in mathematics education. DGEs for 3D-environments (Archimedes Geo3D and Cabri 3D) were designed in Germany and France. In a first study we could show that pre-service teachers with previous knowledge in 2D-systems prefer to work with a real model of a cube instead of the 3D-system to solve certain problems. Furthermore we could find out that previous knowledge in 2D-systems seems to be insufficient to handle the drag-mode in an appropriate way in 3D-environments. In a second study we introduced the students to the special software before the investigation and distinguished different dragging modalities during the solution processes of two tasks.

THEORETICAL FRAMEWORK

During the last three decades, several 2D-Dynamic Geometry Environments (DGEs) have been created to enrich and further the learning process in the mathematics classroom. The most popular DGEs are Cabri-géomètre, GEOLOG, Geometer's Sketchpad, Geometry Inventor, Geometric Supposer and Thales. In Germany, Euklid-DynaGeo, Cinderella, GeoGebra, Geonext and Zirkel-und-Lineal are popular, with Euklid-DynaGeo being the most widespread software in German schools. DGEs are powerful tools, in which the user is able to exactly construct geometrically, discover dependencies, develop or refute conjectures or to get ideas for proofs.

DGEs are characterised by three central properties: the ”drag-mode”, the functionality ”locus of points” and the ability to construct ”macros”. The drag-mode is the most important feature available in these environments, because it allows to introduce movement into static Euclidean Geometry (Sträßer 2002). It is possible to drag basic points (points which are neither intersection points nor points with given coordinates). During this dragging process, the construction is updated, according to the construction commands which were used in the drawing. To the user, it looks as if the drawing is respecting the laws of geometry while the dragging process is in progress.

2D-DGEs are one of the best researched topics in mathematics education and especially within the PME-group (Laborde et al. 2006). For example, we find research on ”DGE and the move from the spatial to the theoretical” (Arzarello et al. 1998, 2002) or ”construction tasks” (Soury-Lavergne 1998). Noss (1994) has shown that beginners have problems to construct drawings, which are resistant to the drag-mode and it is reported that for pupils there exist two separate worlds, the theoretical one and the world of the computer. ”The notion of dependency and functional relationship” (Hoyles 1998 and Jones 1996) is another interesting theme and it has been shown that pupils have heavy problems in understanding the notion of dependency. They have to be encouraged to use the drag-mode to support the understanding of the spatial-graphical and the theoretical level, serving as a tool for externalising the notion of dependency. Several researchers showed that students do not use the drag-mode spontaneously and they have to be encouraged to do it. Most of the students are afraid to destroy the construction by using the drag-mode and they do not like to use the drag-mode on a wide zone (Rolet 1996 and Sinclair 2003). Arzarello and his group elaborated a hierarchy of several dragging modalities, which were linked to ”ascending” and ”descending” processes and reveal students’ cognitive shifts from the perceptual level to the theoretical one (Arzarello 1998, 2002 and Olivero 2002). There is a great variety and number of research reports concerning the use of the drag-mode in proving and justifying processes (for example Jones 2000 and Mariotti 2000). Other fields of study were ”the design of tasks” (Laborde 2001), ”the role of feedback” (Hadas 2000) and ”the use of geometry technology by teachers” (Noss, Hoyles 1996).

THE FIRST STUDY IN 2007

In the following we will give a brief summary of the research design and the results of our first study. For details see Hattermann, 2008. In July 2007, 15 pre-service teachers with previous knowledge in Euklid DynaGeo (2D-DGE) took part in our investigation. Some groups worked with Archimedes Geo 3D and others with Cabri 3D, their actions on the screen and their discussions and interactions were recorded by a screen-recording software called “Camtasia” and a webcam. We used a qualitative approach to get ideas about students’ behaviour in 3D-DGEs. Some important research questions were the following:

- Do the students use spatial constructions like spheres or do they prefer elements from plane geometry? (Task 1)
- What are the preferred tools to work with (paper and pencil, real model, imagination, DGE) to work with? (Task 2)
- Do students use the drag-mode to validate a construction and to find solutions to problems? (Task 1 and 2)
- How do participants behave in 3D-environments and how do they use the drag-mode? (Task 1 and 2)

Task 1 and Results

The first task was: “Construct a cube without using the existing macro!” Five of seven groups constructed the cube. The Cabri groups needed between 20 and 25 minutes to construct the cube, whereas the Archimedes groups needed about 40 minutes. Different groupes tried to utilise transformations as reflections or rotations. While the realisation of a reflection is quite easy in Cabri, rotations seem not to be easy to handle without any instructions. In the Archimedes environment students had problems with every transformation. The majority of the students used the drag-mode to validate their construction only on demand. This result is comparable to the results obtained by Rolet and Sinclair who worked with school children in 2D-environments. Our probands preferred to measure several segments of the cube instead of dragging a basic point. During the construction, elements from plane geometry (circles, segments, straight lines) were preferred. Some groups used spheres to construct intersection points or to construct equidistant segments, but the majority of the groups worked with circles.

Task 2 and Results

The second task was: “A student affirms: The slice plane between a cube and a plane can be:

- an equilateral triangle
- an isosceles triangle
- a right-angled isosceles triangle
- a regular hexagon.

Construct (with the help of the function “cube”) a cube, check the student’s affirmations and justify your results!”

Every group tried to find validations for their conjectures with the help of the real model, the utilisation of the real model prevailed the use of the computer environment. Students preferred ”the old strategy” to examine the cube and to try to imagine the intersection figure. The software was used to validate the conjectures, which were mostly generated outside the software environment. The students defined a plane with the help of three fixed points, so that no dragging was possible. Furthermore, the drag-mode was not understood and it is not sure, if these students did not understand it in the 2D-case or if they could not negotiate it to the 3D-environments. The possibilities of the drag-mode were not understandable to most students. They did not use the drag-mode in an expected manner (to use draggable points on an edge of the cube to define the intersection plane and to drag it to scrutinise different intersection figures). The approach of one group could illustrate this result: The students defined many fixed points on every edge of the cube and defined a plane with the help of three points. After verification, they deleted the plane and constructed another one with the help of other points. Only in exceptional cases the drag-mode was used and more often than not in a manner that a controlled dragging of the plane was impossible, which is the case when students used three arbitrary points in space to define the intersection plane. Students’ statements support the assertion that the “drag-mode” was not understood and previous knowledge in 2D seems to be insufficient to handle 3D-systems!

THE SECOND STUDY IN 2008

Methodology

Our second study took place in February 2008 at the University of Giessen and 15 pre-service teacher students participated in it. The participants had previous knowledge in Euklid DynaGeo (the most widespread 2D-DGE in Germany), but their experiences with DGEs were less than those from students who participated in our first study, because of changes concerning the content of different lectures following new study regulations. There were seven groups (six groups of two students and one group of three students). Three groups worked with Archimedes Geo3D while four groups utilised Cabri 3D to solve the given tasks. Each group worked in a separate room, the actions on the screen were recorded by utilising the screen-recording software "Camtasia". Furthermore, a webcam and a microphone were used to record students' voices and interactions.

In our second study we tried to create an environment in which we could observe different dragging modalities. Due to the results of our first study we opted for an approach with a preparation session in which students were introduced to the special software environment and were encouraged to use the drag-mode. Both groups were taught in:

- dragging basic points in 3D-space in the special software environment with the help of the keyboard
- the distinction between basic points, semi-draggable points and fixed points
- the construction of a midpoint of two points
- the construction of a **"**perpendicular plane**"** to a straight line through a given point beyond the straight line
- the construction of a "perpendicular line" in the x-y-plane to a given straight line in the x-y-plane through a given point , beyond the straight line
- in the construction of a circle in an arbitrary plane, devoid of the x-y-plane, with a given centre and through a new point on the plane
- in reflecting the circle on an arbitrary point devoid of the circle's centre
- in constructing a plane which contains a given straight line
- in constructing a plane with the help of three points in such a way that one of these points can be dragged on a straight line

Archimedes-groups were especially introduced to the utilisation of transformations which is quite complicated in this environment. After the first introduction students were urged to solve five task which forced students to use the drag-mode. Here, we followed suggestions from the Centre informatique pédagogique (CIP 1996) for 2D-environments and adapted the ideas to our 3D-environment. There were five files and every file contained a special task. Every task consisted of a body and one or several yellow points which had been constructed by the researchers before. The task was to find hypotheses concerning the construction of the yellow point(s) by dragging a special point which was marked in blue colour. With the help of these preparation tasks, we intended to weaken students' constraints to use the drag-mode and to encourage them. Because of the domination of the real model compared to the software environment in our first study, we decided to forbid paper and pencil and not to provide a real model of the cube.

In our preparation session, we tried to provide students with competencies to solve the tasks which were given in our study without giving them exact hints. So we broached the issue of constructing a perpendicular line to a straight line through a given point on a special plane without mentioning that this construction could be useful to construct a cube. For another example, students had to construct a plane in such a way that one point of this plane could be dragged on a straight line. The idea behind was to show students how to construct a "draggable plane" without telling them that it could be an appropriate way to scrutinise different intersection figures of a plane and another body by using three defining points of the plane on appropriate segments of the body, which seems to be a reasonable way to solve our second task in the study.

Research questions

First of all we are interested in the general behaviour of our students in a 3D-environment; especially we looked for differences in students' behaviour during the solution process of different tasks compared to the first group in July 2007 which had no preparation session. Are there important differences among the two DGEs? Because of the importance of the drag-mode in DGEs, we want to know more about the utilisation of it, especially we are interested in different dragging modalities in 3D-environments. Do students use the drag-mode to validate their construction in task one (construction of a cube)? A validation of the construction with the help of the drag-mode assumed, how do they use it? Are they more "courageous" than their predecessors in July 2007 and do they use the drag-mode on a "wider zone"? What are the preferred tools to construct a cube? Is one preparation session enough to get students familiar with a 3D-DGE in such a way that elements like spheres or 3D-reflections will be used to construct a cube or do constructions like circles (elements from planar geometry) prevail the construction?

Do students use the drag-mode to discover different intersection figures of a cube and a plane or do they try to avoid the utilisation of the drag-mode in task two? Is it possible to identify different "ways of dragging"? What solving strategies are preferred by students who do not possess neither a real model of a cube nor a paper and pencil environment?

Task one and Results

We used the same task as in our first study in July 2007:”Construct a cube without using the existing macro!”

Every group constructed the cube. The Cabri-groups needed 17, 19, 26 and 41 minutes for the construction, whereas the Archimedes-groups needed 34, 37 and 45 minutes. Furthermore every group utilised the drag-mode to validate their construction and two Cabri-groups did it in a “courageous way” so to say, they used it on a wider zone. One Archimedes-Group was very careful by dragging basic points. Every group was very happy by observing the invariance of the constructed cube under dragging and jubilation and pleasure were recognisable in nearly every group. This fact shows that dragging can motivate and emotionally affect students which underlines the importance of this feature.

By comparing the periods of construction it seems as if Cabri-Groups work faster. In our first study we came to the same statement and argued that one reason for this could be the “base plane (x-y-plane)” which exists in Cabri. In Archimedes this plane has to be constructed first. We can’t support this hypothesis with our actual data, because during the preparation session the construction of the x-y-plane in Archimedes was mentioned and every Archimedes-group had no problems to construct it in a short time not exceeding 3 minutes.

No group tried to construct the cube with the help of spheres, only circles, planes and perpendicular lines were used to construct cube vertexes. An explanation for this result lies in the preparation session, in which circles, but no spheres were explicitly mentioned.

One Archimedes-group utilised reflections on a plane and reflections on a straight to construct cube vertexes. One Cabri-group utilised the function of a parallel plane to a given plane but furthermore no reflections were used by students. In our first study no Archimedes-group used reflections to construct the cube. Due to the fact that “transformations” are not easy to handle without instructions, this fact was not surprising to us. After an introduction in defining and utilising transformations in Archimedes, one of three groups used “reflections”, but the size of the sample seems to be too small to interpret this fact in more detail.

Besides we observed students who had problems with “parent-child-relations” (see also Talmon 2004). Several situations occurred, which prove that dependencies of construction objects are not understood completely. Some groups did not understand that objects disappear by deleting an object on which they depend on.

Furthermore we could identify several dragging modalities in 3D-environments. Students used the drag-mode in our first task to

- validate the construction at the end of the construction process.

- see that there are only two draggable points (the points that define the first edge of the cube) and to see that the other points are fixed.
- find out the function of a semi-draggable point on the edge of the cube that had been constructed before. (Students forgot for what reason they had it constructed)
- adapt the length of a segment to the measure of the first edge. (students did not really construct a cube in this attempt, they created a cube which was not invariant under dragging)
- find out more about the degrees of freedom of draggable points, for instance to scrutinise if points are draggable on a plane or only on a straight line.
- find an error in the construction. (Actually the construction was correct, only one point was wrong and this fact was discovered by dragging)

Task two and Results

The second task was changed compared to the version used in July 2007. Task two was the following: “Construct with the help of the function “cube” a cube and try to find by experiment all Polygons (n = 3, 4... n = number of vertexes) which exist as intersection figures between the cube and a plane.” The second task was changed slightly in comparison to the first study, because we intended to further the need for the utilisation of the drag-mode. In the first study we gave four intersection figures and asked students to confirm or refute our statements, whereas the assignment is more open in our second study. We hoped that trying to discover new intersection figures would motivate students and moreover we tried to create an environment in which dragging could help students to find solutions. Finally we intended to observe and distinguish different “ways of dragging” during the solution process.

Except of one group, everybody found the equilateral triangle and the isosceles triangle as an intersection figure. Approximately the half of the participants mentioned an arbitrary triangle as intersection figure, whereas only one group could find a parallelogram. The rectangle and the square were the easiest figures which were found by every group. Half of the groups found the trapezoid as intersection figure, whereas the other participants found it was well, but did not identify this quadrilateral as a trapezoid. Nobody looked for an isosceles trapezoid. Three groups found a pentagon, four groups found a hexagon and four groups found the regular hexagon. There were groups that found the hexagon and not the regular hexagon and vice versa.

During the solution process we observed different dragging modalities. Students used the drag-mode by

- defining the intersection plane by one point on an edge of the cube and two vertexes.

- choosing two points in a Cabri-environment to define the plane (now a plane appears) and to observe the behaviour of this plane by moving the cursor on the screen. (a special type of dragging only available in Cabri-environments)
- defining three points on different edges of the cube to define the plane.
- using three arbitrary points in space to define the intersection plane.
- defining one draggable point on a straight line that is defined by two vertexes of the cube and to use two other points in space to define the plane.

Students used the drag-mode to:

- find out the function of a special point which had been constructed before. (a point was used to define a plane for example)
- vary the volume of the cube so that the intersection points between the cube and the plane become visible (which is not always the case).
- identify new intersection figures.
- get an idea how to construct the intersection figure afterwards with the help of fixed points to define the plane.
- identify more special figures/more general intersection figures from an existent figure. (find an equilateral triangle from an arbitrary triangle or vice versa)
- scrutinise if there are intersection figures with more than 4 vertexes. (with the special type of dragging in Cabri)
- move the cube, instead of varying the plane, to scrutinise different intersection figures.
- identify draggable and non draggable points.

It is really worth mentioning that we could observe happiness in every group by realising different intersection figures with the help of the drag-mode. “Wow” or “that’s really great” are only two short examples that underline our affirmation.

Conclusion

We succeeded in our second study to get the probands more familiar with the special DGE and to observe different dragging modalities in task one and two. There are still situations in which students utilised the drag-mode very careful and not on a wider zone, but the majority of our participants utilised the drag-mode to validate and to discover in a “courageous” manner without hesitation. So we claim that it is possible to prepare students in an appropriate time to use the drag-mode in 3D-systems and to encourage them.

For a classification of different dragging modalities it will be interesting to categorise them theoretically and to analyse the “instrumental genesis” of the drag-mode according to Rabardel’s theory (Rabardel 1995). It will be an exciting task for further research to observe the progress of the utilisation of the drag-mode. It should be possible to define different theoretical stages in the utilisation of the drag-mode from a "beginner's stage" which will be characterised by nearly no dragging or careful dragging up to an "expert's stage".

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3D GEOMETRY AND LEARNING OF MATHEMATICAL REASONING

Joris Mithalal, PhD student

Laboratoire d'Informatique de Grenoble (LIG)

Didactique, Informatique & Apprentissage des Mathématiques (DIAM)

Université Joseph Fourier, Grenoble, France

Teaching mathematical proof is a great issue of mathematics education, and geometry is a traditional context for it. Nevertheless, especially in plane geometry, the students often focus on the drawings. As they can see results, they don't need to use neither axiomatic geometry nor formal proof.

In this thesis work, we tried to analyse how space geometry situations could incite students to use axiomatic geometry. Using Duval's distinctions between iconic and non-iconic visualization, we will discuss here of the potentialities of situations based on a 3D dynamic geometry software, and show a few experimental results.

In mathematics education, resolving geometry problems is a usual way of teaching *mathematical proof*, and plane geometry is mainly used.

Nevertheless the students often focus on the properties of *drawings* — which are physical objects — instead of *figures* — the theoretical ones. In this case they may solve geometry problems by using empirical solutions, based on their own action on the drawing: *One can read the property on the drawing.* That is why using drawings as regards plane geometry is very confusing for many of them: since they are able to *see* results on the drawings, since they can work easily on it, mathematical proof seems to be useless, and may appear as a didactical contract effect (Parzysz, 2006).

On the contrary, in space geometry, it seems to be much harder for them to be certain of a visual noticing, and they may need new tools to study representations and to solve problems.

Our hypothesis is that it is possible, with specific situations, to make the students use tools concerning theoretical objects: working on figures, using geometrical properties… In order to control these new tools, mathematical proof is a very useful process the students can use to solve problems. This is why we assume that 3D geometry could be very helpful for proof teaching.

Nevertheless, formal proof is a complex process that not only involves hypothetico – deductive reasoning, but also (for instance) specific formal rules (Balacheff, 1999) we will not study here. Therefore, we will only focus in this paper on the first hypothesis we mentioned.

We will present here a preliminary study in order to illustrate and test our theoretical hypothesis.

THEORETICAL FRAMEWORK

Resolving problems of geometry

As it is said in Parzysz (2006):

> The resolution of a problem of elementary geometry consists of the successive working with G1 and G2, focusing on the "figure". The figure has a central part in the process: even if it is very helpful in order to make conjectures, it may be an obstacle to the demonstrating process, as the pupils don't know how to use data because of the "obviousness of the visual phenomenon".

Parzysz refers to Houdement&Kuzniak's geometrical paradigms, in so far as G1 is a "natural geometry" — where geometry and reality are merged — and G2 is a "natural axiomatic geometry", an axiomatic model of the reality, based on hypothetico-deductive rules (Houdement, Kuzniak, 2006).

As we can see, demonstrating is really meaningful when working with both G1 and G2, but the sensitive experience may encourage the pupils to work only with G1. In order to describe more precisely what can be this *sensitive experience*, and the ways it is related to using — or not — G2, we chose to use the distinctions that Duval (2005) makes between the different functions of the drawing, and the different ways of seeing it.

A first way of using representations is the *iconic visualization*: in this case the drawing is a true physical object, and its shape is a graphic icon that cannot be modified. All its properties are related to this shape, and so it seems to be very difficult to work on the constitutive parts of it — such as points, lines, etc. Then, the drawing does not represent the object that is studied, it *is* this object, and the results of geometrical activities inform on physical properties.

The other way is the *non-iconic visualization*, where the figure is analysed as a theoretical object represented by the drawing, using three main processes:

Instrumental deconstruction: in order to find how to build the representation with given instruments.

Heuristic breaking down of the shapes: the shape is split up into subparts, as if it was a puzzle.

Dimensional deconstruction: the figure is broken down into *figural units* — lower dimension units that figures are composed of —, and the links between these units are the geometrical properties. It is an axiomatic reconstruction of the figures, based on hypothetico-deductive reasoning.

These different possible ways of using the drawings lead us to two important consequences.

On the one hand, using G2 makes no sense with only iconic visualization, as geometry problems concern nothing but the drawings to the student's eyes.

On the other hand, carrying out the dimensional deconstruction means isolating subparts of the drawing and, at the same time, describing how these subparts are linked: this last part has no sense when using only G1. Therefore this operation implies a more axiomatical point of view, and the figure — described by the dimensional deconstruction — is likely to be used.

Finally, we assume that dimensional deconstruction would become an efficient tool if the iconic visualization weren't reliable any longer, as the pupil would have to make up for the lack of information in order to solve geometry problems. Using graphic representations is much more complex in space geometry, and then it seems to be an appropriate environment for the teaching of axiomatic geometry.

3D geometry

Using physical representations is very different in space geometry: there are various ways of representing figures, such as models or plane projections, and each kind of representation has specific properties and constraints. As the physical models are too restricting — for instance, adding new lines is generally impossible, and constructing models takes much time —, cavalier perspective representations are generally used. Then, visual information is no longer reliable: for instance, it is impossible to know whether two lines intersect or not, or whether a point is on a plane, without further information.

So in space geometry iconic visualization fails, and it is necessary to analyse the drawings in other ways. The problem is that using drawings is generally too difficult for the pupils. Chaachoua (1997) mentions that this involves the students' interpretation, based on their mathematical and cultural knowledge. They have to break down the drawing into various components, so that they can imagine the shape of the object. In fact, they would have to carry out dimensional deconstruction *before* any visual exploration. Therefore they are unable to understand that iconic visualization is not sufficient to solve geometry problems, as they only think that they *see* nothing.

Using 3D geometry computer environments may balance these difficulties, since the students could get more visual information, for instance by using various viewpoints as if the representations were models. It has to be noticed that, even in this kind of environment, visual information is usually not reliable, so that iconic visualization remains inadequate to solve geometry problems.

Hypothesis about Cabri 3D

With Cabri 3D, the user can watch the representation as if they were models. It is possible to adjust viewing angles by turning around the scene, to look at the drawing from various viewpoints, and then to be more easily conscious of the visual issues. For instance, it becomes possible to see that a point belongs to a plan, when the point *visually* belongs to it. Actually the user can get visual information to determine the shape and some properties of the figures, but generally this information is not sufficient to carry out geometrical works. For instance, as the representations are not infinite in Cabri 3D, two secant lines could have no intersection point on the screen, then it would be impossible to determine visually whether these lines are secant or not. Some operations are almost impossible too, like moving a point to reach a given line with no other tools than visual perception.

Then, the feedback from a Cabri 3D - based *milieu* — as described in Brousseau (1997) — may emphasize that, even if visual information is available, this information is partial. A Cabri3D drawing does not permit to *see* all the specificities of the object the student has to study – which is clearly not the drawing itself.

It seems that a problem any student would have to deal with, when using Cabri3D, is “How can I get information from the drawing, and how may I use it in order to deduce information I cannot see, and solve geometry problems?”. We showed that there are two main kinds of answers: the iconic visualization based ones, and the non-iconic visualization based ones.

Our first hypothesis is that with Cabri 3D it is much easier for the students to get information about the drawings, and then to start a research process, even if they only use iconic visualization. This research process may evolve because of the dynamic geometry software properties of Cabri3D.

Cabri 3D not only produces representations, it is a dynamic geometry software. In this way it is possible to use *hard* geometric constructions: these drawings are based on geometric properties, and keep it when the user drags a part of it. As an example, a hard square remains to be a square — with different size — when one of its vertexes is dragged. Therefore, the students may assume that the reason of simultaneous movements of figural units is the relation between them: if a point moves when another one is dragged, it may seem that they are *linked*, in a way that has to be elucidated by the students.

We can guess that this point is stressed in 3D dynamic geometry situations, since other visual information is generally not reliable: one can be sure of the simultaneous movement of two figural units, even if it can be quite difficult to determine *how* these units are linked. These links are in fact invariant properties when points are dragged, and then direct results in Cabri3D of geometrical properties (Jahn, 1998).

Our second hypothesis is that with dynamic geometry it is possible to stress the inefficiency of iconic visualization, and to support experimental studies of the properties of the figure. Therefore dimensional deconstruction and axiomatic geometry would become very efficient tools for the students to design research processes, to study a given representation and to solve geometry problems.

Nevertheless, these theoretical tools are not sufficient: any experimental work in Cabri 3D has to involve Cabri 3D's tools. Therefore we have to study their role and the way they could interact with the theoretical ones.

First, many tools are very linked to visual perception: changing viewpoint tools, drawing and measuring tools. If they are not used with other tools, there is no need for the student to control her/his work with G2. S/he can measure drawings, watch their shape and construct objects as *soft*, and not hard constructions. When a part of such a drawing is dragged, the shape changes and so do the geometric properties the user can see. Then the feedback from Cabri 3D invalidate this kind of construction to the user's eyes (Laborde, Capponi, 1994).

Secondly, other tools are more strongly linked to a theoretical control of the constructions: construction primitives — intersection, parallel, perpendicular, tetrahedron, etc. — and transformations. Even if using axiomatic geometry is not necessary to control the use of these tools, an empirical control may be very difficult in many situations (for instance, in order to use a transformation, the user generally has to choose the values of several arguments before any visual control). So using G2 would become an economical way of controlling it. Furthermore, these tools would be very helpful for the process of instrumental deconstruction, as they are designed with axiomatic definitions. Actually, for this reason, instrumental and dimensional deconstructions would be very linked in this case.

Eventually, we have to point out that the designer of a situation (teacher, researcher…) can choose the toolset available in Cabri 3D. This is a way for him to delete specific tools in order to design feedbacks. For instance, if the students have to construct hard squares, there is no feedback about the hardness of constructions when using the "square" tool. Therefore choosing the available toolset is often a very important choice for this *didactical variable*, to make strategies inefficient or impossible.

Then, our third hypothesis is that in some specific situations, with a specific Cabri 3D toolset, it is possible to provoke a particular instrumental deconstruction, strongly linked to dimensional deconstruction.

Research problem

As a consequence of our theoretical framework, it is now possible to make the problem mentioned in the introduction clearer and more accurate: is it possible to design adidactical situations with Cabri 3D that make iconic visualization inefficient and in which dimensional deconstruction can be a tool to analyse figures and solve problems? Then we have to wonder whether using dimensional deconstruction could be liable to make the students using G2.

The following example is a situation we designed in order to test our hypothesis, in which a student has to analyse a Cabri3D-drawing in order to explain to another student how to construct the same object with Cabri 3D.

AN EXAMPLE OF A RECONSTRUCTION SITUATION

Methodology

We used a qualitative approach to analyse the students dealing with this task. We refered to our theoretical study in order to distinguish different strategies they were likely to use. It was possible to foresee how they would analyse the drawings, as shapes or as geometrical constructions... Moreover we had to analyse how they design their construction strategies. For instance, anticipating the properties of the object constructed would reveal G2-based strategies. We will only detail below the three main kind of strategies we distinguished.

In order to analyse the students' work, we used a screen-recorder software (Camtasia), microphones, and a video camera. Then we could observe at the same time their dialog, their gestures (for instance to describe physical objects), and the way they used Cabri 3D.

The situation.

This situation involves 10th French graders (15 to 16 year-old students), working in pairs. Each student works on a computer. The first one (S1) has to analyse a model, a Cabri3D-drawing, and describe orally to the second student (S2) a way of reconstructing it. Using S2's computer is forbidden to S1, and S2 cannot see S1's screen.

There are four distinct phases, from the simple to the complex one (see Fig.1): first a prism with a rhombus as a base, and then are successively added its symmetrical with respect to a vertex, an edge and a lateral face. All these prisms are constructed from three directly movable points: a and b are in the base plane, and c is on the line perpendicular to the base plane at point O (the centre of the bottom face of the prism). All the other points are constructed using symmetries, so that the constructions are robust ones.

S2 is given a file with the three points, a, b and c, and the two students have to validate their constructions by themselves. The only condition is that the behaviour of the new object has to be the same as the model's one when point a, b or c are moved.

S2 doesn't see the prism and the polyhedron tool is not available, so it is much harder to solve empirically the three last problems by constructing symmetricals of the first prism.

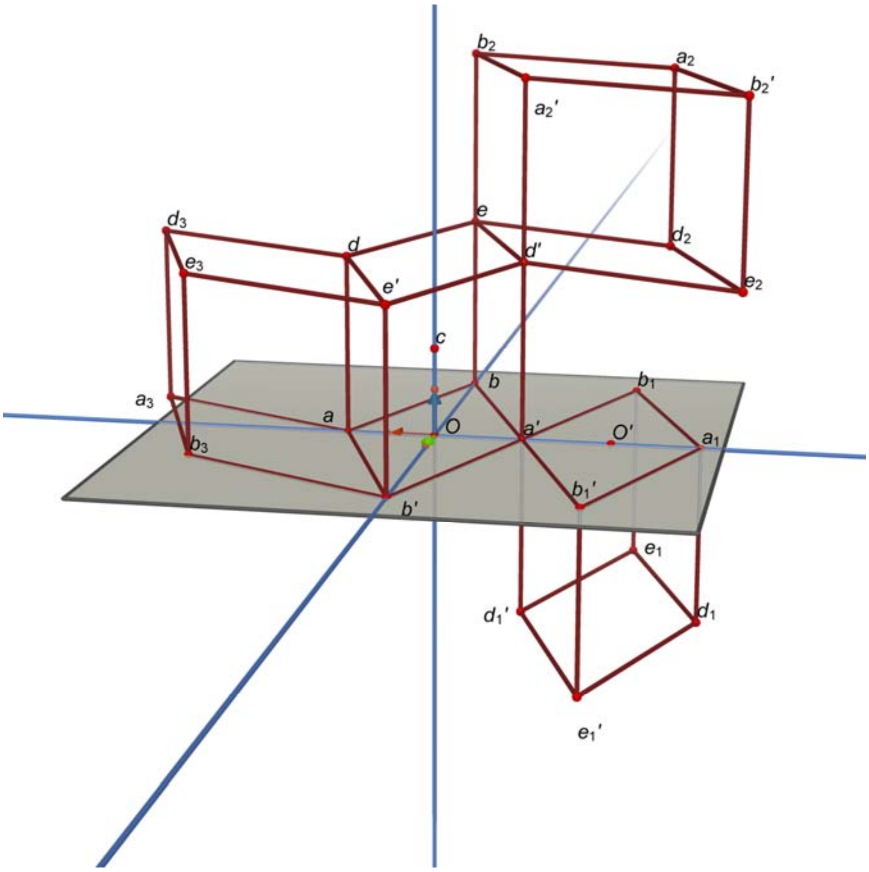

Fig. 1: Figure to analyse and reproduce in phase 4 (in previous phase, parts of the figure have been reconstructed)

Three strategies

First, if they worked using only G1, they would analyse the shapes and sizes of the models, and try to reproduce it by creating points and dragging it to the right positions. This is very difficult in a 3D space represented in 2D, and we can guess that construction primitives may be used as stands on which a visual control of the positions is possible. This is a basic strategy, and it fails in Cabri 3D whereas it wouldn't in a paper/pencil environment. We call it R1.

The second strategy (R2) is based on the use of construction primitives controlled by knowledge about "basis configurations" (Robert, 1998) learnt before. For instance, point O may be recognized as the centre of symmetry of the bottom rhombus not because a and a' seem to be symmetrical with respect to it, but because the student already know that the "centre" of a rhombus is its centre of symmetry. Therefore the students may use locally plane transformations (on some planes). But in space, as they have no previous knowledge about symmetry in a prism, their strategy may be similar to R1. We expect that in this case, in the model analysis phase and in the interaction with S2 phase, S1 may focus at the same time on geometric properties and on size information. This strategy does not necessarily require dimensional deconstruction. The result of it is a partial failure, as the dynamic properties exist in planes, but not in space.

The third strategy (R3) may be based on transformations. In this case, we assume that the student use axiomatic geometry and dimensional deconstruction, then we can guess that their analysis would focus on invariant properties when they drag points, and their reconstruction strategy would be designed in order to reproduce these properties.

Experimental results

We experimented this task with three pairs of 10^{th} French graders, who had been just introduced to Cabri 3D before. Our following analyse will mainly focus on the "reconstruction phases", and not on S1's analysis of the drawings.

First of all, it seems that the students could get information about the drawings by manipulating it. They were able to determine, visually, shapes and basic physical properties, and to try to find a solution to the problem. For instance, the Group 3 students only used iconic visualization, and they could construct the prism shape – but a soft construction, based on the length of the edges. They tried something, and their failure was not the consequence of the too high complexity but was linked to the expected properties: some points *"don't move"*.

Secondly, all the students realized that iconic visualization was not sufficient to carry out the expected construction. We have to distinguish to main cases.

Groups 1 and 2 first used only R1, but they realised that this strategy was no longer efficient in 3D geometry. As they were able to use – more or less easily – non-iconic visualisation, they tried other strategies and could reproduce the dynamic properties. It has to be noticed that they used R2 and R3 because it was easier that R1, and not in order to make *hard* constructions (even if this was a consequence).

On the contrary, at the beginning, Group 3 students were not able to use anything but iconic visualisation. They constructed the first prism with R1, which led them to a failure: the points "didn't move". Iconic visualisation couldn't help them to analyse this:

S1: Try to make the point move

S2: I can't, there is no line [on which the point could move]

Then they started to use iconic and non-iconic visualisation at the same time, depending on their aim. For instance, they first tried to make b', b_1 and b_1' while dragging b, but didn't care about a, a'... They kept constructing a, a', a_1, etc., by measuring lengths, but constructed b_1 and b_1' by using geometrical properties, such as “parallel”, instead of adjusting positions. This second case underlines that using non-iconic visualisation can be strongly linked to the dynamic properties of the drawing.

Eventually, we have to point out that the students didn’t use easily dimensional deconstruction, and then they first tried to use it as little as possible. For instance, it seemed to Group 2 students that ded’e’ and ed_2’e_2’d’ (see Fig. 1) were linked, and that (ed’) had something to do with this link: “*a rotation*”. They tried to use the tool without any further analysis (basic instrumental deconstruction), and couldn’t succeed. Then, they analysed more precisely the link, and discovered that they had to use “*symmetry*”. Actually, as instrumental deconstruction was not precise enough, thay used dimensional deconstruction in order to control more precisely the way they used the tools.

CONCLUSION

Finally, our experimental results have a global consistency with the three hypothesis we mentionned.

The students used the representations as if they were models, and could get information from it. Even if they wanted to draw shapes, without any dynamical properties, they were able to get enough information by *looking* and *measuring* the models. Moreover, we could observe that, even to draw shapes, non-iconic visualization led them to more efficient strategies (Groups 1 and 2).

Nevertheless, because of the dynamic geometry, this process was inefficient, and they had to find a way of reproducing dynamical effects. With this new research process, they had not only to use iconic visualization but to find something else. Depending on the students’ knowledge, most of them tried to use dimensional deconstruction and an axiomatical point of view, as the most efficient strategy – efficient for analysing, giving oral information, reconstructing, arguing... In every group, the strategies used by the students evolved and dimensional deconstruction was more and more involved, so that they were able to give an interpretation to dynamical effects.

It seems that Cabri3D’s tools were very important in the evolution of strategies. Using of transformations appeared to be a way of solving the problems, but an empirical control was very difficult in most cases. Then, the students changed their strategies, and tried to find new ways of controlling it, by using dimensional deconstruction.

Therefore, these results give us informations about our research question: iconic visualisation failed, and dimensional deconstruction was necessary to solve the problem. Moreover, even the weakest students started using dimensional deconstruction, whereas they were unable to do so at the begining of the exercise. Then we could ask two new questions, more accurate. One the one hand, how did dimensional deconstruction appear, and how is it related both to the task and to instrumental deconstruction? On the other hand, we will have to study wether using dimensional deconstruction is liable to make the students use G2 in geometry, and not only in 3D geometry.

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IN SEARCH OF ELEMENTS FOR A COMPETENCE MODEL IN SOLID GEOMETRY TEACHING. ESTABLISHMENT OF RELATIONSHIPS

Edna González; Gregoria Guillén

Departamento de Didáctica de la Matemática. Universitat de València. Spain

* Scholarship holder CONACYT, Mexico.

ABSTRACT

In this paper we present part of the analysis of a Teaching Model for the geometry of solids of an initial Education Plan for elementary school teachers, and its implementation in the University School of Teaching of the Universitat de València, Spain. We have focused our attention on how the establishment of relationships among geometric concepts have been worked on. For this analysis we considered theoretical contents related to geometric contents (concepts, mathematical processes and different types of relationships). This study is part of a more extensive work that tried to elaborate the competent conduct features for a teacher teaching solid geometry in elementary school.

PRESENTATION

This work is part of a more extensive research project which uses as a methodological framework the theory of the "Modelos Teóricos Locales" (MTL) (Local Theoretical Models) (Filloy, 1999). According to Filloy and col. (1999), to be able to take into account the complexity of the phenomena that take place in the educative systems, the MTL incorporate several interrelated theoretical components: 1) Competence Model; 2) Teaching Model; 3) Cognitive Processes Model, and 4) Communication Processes Model. Our work is focused on the first component in relation with the training process of elementary school teachers in the subject of solid geometry.

De Ponte and Chapman (2006) point out that this research line has given priority to the analysis of teachers knowledge or practice paying less attention to the analysis of the programs for their training. In our work we analyze a solid geometry training Program for elementary school teachers and its putting into practice; we want to establish some elements for the Initial Competence Model (ICM) in relation with the training of elementary education teachers in the geometry of solids. In previous papers we have presented elements of this competence model that show a competent conduct for teaching mathematical processes related with describing, classifying, generalizing and particularizing. In the present paper we focus on the elements related to the establishment of relationships among geometrical contents.

BACKGROUND AND FRAMEWORK

The analysis we present in this paper is part of a more extensive work - González (2006)[1]-, which had the purpose of elaborating the elements for an ICM that can be used as a reference to interpret the teaching models proposed for teaching solid geometry in training programs for elementary school teachers. This work belongs to a project that aimed for the creation of a "Virtual Library"[2] that could help to teachers' permanent education.

In previous works (González and col. 2006, 2008; González, E. and Guillén, G. 2008) we have presented some results of the analysis. To group these results we have followed the distinction made by Climent and Carrillo (2003), who take into consideration teacher's knowledge and distinguish as different components the mathematical content knowledge (in our case contents *of* and *about* geometry) and the knowledge of the subject for its teaching.

In previous papers above mentioned we refer to results that have to do with the contents of "solid geometry" related to mathematical processes of classifying, describing, generalizing, and particularizing. We show how the attempt of organizing the surrounding objects and their construction, by means of different procedures, provides very rich contexts to develop these mathematical processes. We also present some of the reflections encouraged by the teacher concerning the learning process of both children and teachers, questions having to do with preparing the lesson, are related to the use of language, or the way to respond to the appearance of misconceptions.

The observations we present in this paper belong to the first group of contents *of* and *about* geometry, and complete the study; these observations refer to relationships among geometric objects of the same and different dimension; that is, relationships among solids, among their elements or among plane and space elements.

As we advanced in the presentation, we follow the Theory of the MTL as experimental methodological framework. We have commented that in this Theory four interrelated theoretical components can be distinguished. What differences each component from the others is, among others, the phenomena taken into account

[1] Work carried out to obtain the "Diploma de Estudios Avanzados" (Certificate of Avanced Studies) of the PhD program of Mathematics Education. Universitat de València, Spain.

[2] Project "Procesos de transferencia de resultados de investigación al aula: el caso del bajo rendimiento escolar en matemáticas". Research project, co-financed by the Consejo Nacional de Ciencia y Tecnología (CONACYT-G37301-S) (Nacional Council for Science and Technology). México.

http://www.pernodis.com/ptria/index.htm. In the site dedicated to geometry, section *"Descubrir y matematizar a partir del mundo de las formas"*, chapter *¿Cómo enseñan otros?* we present extracts of the class sessions with the corresponding analysis (http://hipatia.matedu.cinvestav.mx/~descubrirymat/).

regarding the concept subject of analysis. In this work in particular, the ICM includes elements of the knowledge of an ideal person, capable of carrying out tasks related to the teaching of solid geometry at elementary school level. This is, it includes the elements which should be part of the competent conduct of elementary school teachers when teaching the geometric topics regarding solids in their classes. We have already pointed out that the elements commented in this work refer to the establishment of relationships among geometric contents.

When we focus on solids, our theoretical framework is based on the studies made in Didactics of solid geometry (Guillén, 1991, 1997; Guillén and Figueras, 2005), we continue reorganizing these contents as referred to: a) geometric concepts, b) mathematical processes (to analyze, to describe, to classify, to generalize, etc.), c) relations among geometric contents. When we studied how these geometric contents were taught, we also paid attention on how the skills are used (to construct, to modify, to transform) to work the mathematical processes indicated or to develop skills (to communicate and/or to represent forms). The reorganization of the school contents has leaded to organize the observations as related to the teaching/learning of concepts, of mathematical processes, or of the establishment of relationships among different geometric contents. The observations made are detailed in Guillén (1991, 1997). These works take into account, on the one hand, relationships among solids and/or families of solids. These refer to inscription and duality relationships among families of solids, to composition or decomposition relationships, or to inclusion, exclusion or overlapping relationships among different classes established with different classification types (hierarchic partitions or classifications) taking into account several universes and criteria for classifying. On the other hand, we stand out the relationships among the solids elements that can be either of parallelism and perpendicularity or numerical relationships among them. Also were taken into account the relationships among geometric contents of several dimensions that emerge when solids truncate or during the construction of models parting from a plane surface. Moreover, attention has been paid to the establishment of relationships by analogy. In the work of González and Guillén (2006) the inclusion, exclusion or overlapping relationships among families of solids were studied. The rest of types of relationships are the ones that have been taken as reference to organize the observations that this report presents.

The studies above mentioned have been developed taking as a reference the works of Freudenthal (1973, 1983) and others, that have been carried out at the Freudenthal Institute (for example Treffers 1987). These works are the theoretical basis for our concepts over geometry and its teaching, over the relationships among the different geometric contents, and also provides us with information to organize the solids geometry teaching. In this framework one of the aims of geometry teaching is the development of mathematization through mathematical practice.

To carry out the analysis we have also taken as a reference other studies about the appropriate contents for the teachers training plans, emphasizing on the different contents that should be discussed on a reflective level (Shulman, 1986; Climent y Carrillo, 2003; De Ponte y Chapman, 2006; González et. al. 2006).

DATA COLLECTION AND ANALYSIS

To create the MCI, we analyzed the available literature related to the mathematical content analysis and observation of the learning process for mathematical processes and the literature related to teachers' education, this enabled us to elaborate the Theoretical Framework of the work and define the criteria used to analyze the design and implementation of a Teaching Model of the teacher of Teaching with an extensive experience in introducing to the study of geometry having as a support solids geometry.

The work has been developed in several stages. In the first one, we examine theoretical works of the research lines we mentioned in the previous section and the teachers' training plan of the teacher who constitutes the study scope of our work (Guillen, 2000). In a second stage we analyzed the implementation of this training plan.

The data for this experimental study was obtained during the 2005-2006 school year. We attended and took notes of 22 class sessions the training teacher dedicated to solid geometry during the course she gave to a group of students belonging to the foreign language specialty at the University School of Teaching of the Universitat de València (Spain). Each session lasted 50 minutes approximately.

To control all the information that emerged during the teaching, the sessions were recorded in video and audio. These recordings were transcribed and from them, together with the notes taken during the classes, were obtained the extracts to carry out the analysis. These were considered the essential element and were defined taking as a reference the theoretical analyses performed during the first stage. They could be a sentence or a set of sentences that not necessarily had to match the answers or individual interventions of the teacher or of the students.

These extracts were organized in groups as it follows: i) On geometry and its teaching. Student and teacher; ii) On geometric contents; iii) How do some of those students learn? What for?; iv) The class planning; v) Interacting in the class and ... vi) What about language? In Gonzalez et al. (2006) we briefly detail observations related to each of them.

The school contents organization we carried out, mentioned in the previous section, show the distinction we made in the observations we included in group ii). We separated them as follows: ii.1) relative to concepts learning; ii.2) relative to mathematical processes; ii.3) relative to the establishment of relationships. We have already mentioned that in the following section we will refer to group ii.3).

To analyze the corresponding extracts for the establishment of relationships we used, on one hand, the diagram presented by Olvera (2007) and showed in figure 1. This diagram was constructed starting from the characteristics of Van Hiele levels for solid geometry determined in the study by Guillén (1997). On the other hand, in its organization the families of solids and polygons implicated and the relations among flat geometric objects and space geometric objects were taken into account. Also different representations of the solids used as a context were considered and numerical relations were also underlined.

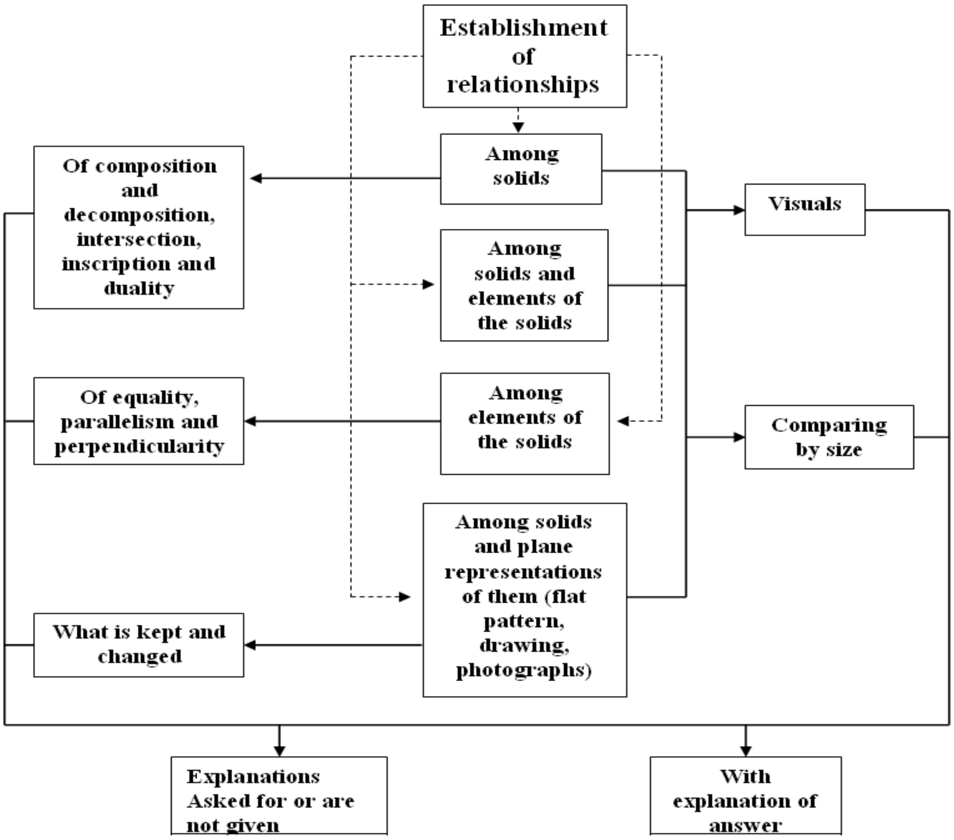


Figure 1

IMPLEMENTATION OF A TEACHING MODEL FOR SOLID GEOMETRY. OBSERVATIONS RELATED TO THE RELATIONSHIPS AMONG THE GEOMETRIC CONTENTS

In Figure 1 we show how the observations of relationships among geometric contents during the implementation of the analyzed training plan are grouped. Following, we present some examples.

Establishment of relationships

The observations that we present in this section have been organized taking into account, on the one hand, the solid families used as a support to develop the activity. On the other one, that the context can also consist of the different representations of solids. It is also necessary to take into account that the relations established could also be numerical.

1. Relations of inscription and duality among regular polyhedrons. When numerical relations are exposed in a table as shown in Figure 2, in which the number of faces, vertexes, edges, order of the vertexes and number of sides of the polygons of the faces have been registered, it leads to the establishment of a wide variety of relationships.

For example, it comes to express that the number of faces of the dodecahedron is equal to the number of vertexes of the icosahedron; or that the number of vertexes of the octahedron is equal to the number of faces in the cube. From this type of relationships, it can be concluded that some polyhedrons can be inscribed in others. For example, the cube can be inscribed in a octahedron in such a way that the vertexes of the cube are in the center of the faces of the octahedron, or vice versa.

	C	V	A	O de V	Forma de C
Tetraedro	4	4	6	3	3
Cubo	6	8	12	3	4
Octaedro	8	6	12	4	3
Dodecaedro	12	20	30	3	5
Icosaedro	20	12	30	5	3

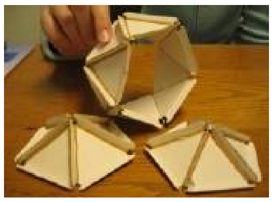
Figure 2

Octahedron inscribed in the cube

There are also relations established among elements of the dual regular polyhedrons when instead of considering models of pairs of dual regular polyhedrons inscribed, compound models are considered, which are intersections of pairs of dual polyhedrons. For example, the cube and the octahedron.

After encouraging students to imagine in a dynamic way how to pass from the inscribed model to the compound model when the size of the inscribed polyhedron is increased, the attention is focused on the fact that the edges of both polyhedrons cut perpendicularly at their midpoint.

2. Relations among regular polyhedrons and other solid families. When trying to analyze regular polyhedrons, they have repeatedly been studied in relation to other families. For example, in the analysis of the icosahedron it is emphasized that it can be seen, on the one hand, as the composition of two pentagonal pyramids of regular faces and a pentagonal antiprism of regular faces or as the fitting of two caps that correspond, each of them, to a pentagonal bipyramid of regular faces, in which one of the pyramids has been opened.



3. Cylinders and Prisms. Cones and pyramids. Immersed in the situation of generating models with different procedures, in first place, the family of straight prisms was introduced through the truncation of a straight cylinder.

For example, questions raise such as: What form do we obtain if we cut perpendicularly the base? How many cuts, perpendicular to the base, should be done for the circle of the base to turn into a 5-sided polygon? What does the cylindrical surface turn into? How are the cylinders obtained with parallel to the base cuts? Can we also obtain oblique prisms? And this problem extends to the establishment of relations between cones and pyramids.

Likewise, comparisons among naive ideas and properties of both families are established. For example, it is pointed out that with parallel cuts to the bases in both families (cylinders and prisms) the shape of the sections is maintained (same form of the bases), and these cuts divide the corresponding solid into other solids with the same form, with the same bases as the original one; and, when adding the corresponding heights, the original solid height is obtained. Immersed in this matter, it is concluded that some prisms can be inscribed in cylinders raising the question of which polygons can be inscribed in a circumference?

4. Comparing cylinders and cones. Prisms and pyramids. When considering a dynamic transformation of one family into another, this transformation is profited to establish relations among the elements of the families of implied solids. For example, when the attention is focused on the transition from a prism into a pyramid, one of the bases of the prism is reduced to a point in the pyramid and it results in the transformation of the lateral faces of the prism into triangles, or that the number of faces in prisms is reduced by one in the number of faces of pyramids, etc.

5. Families of solids and flat shapes. When we focus on counting the elements of regular polyhedrons paying attention to their layout in space, relations are established among this layout and the form of the cuts sections equidistant from opposite faces, vertices or edges. The study is completed with the determination of the different types of planes of symmetry and axes of rotation of each regular polyhedron and the number of planes and axes of each type.

In a context of truncation in cylinders, cones, spheres, prisms and pyramids the relations among the direction of the cut and the form of the sections are established. The process is also considered in a dynamic way; that is, it starts with the observation of a section shape and this is compared with the other sections obtained by parallel cuttings done to the original.

6. Different representations of the solids as a starting point. This situation enables setting relations among different representations or among the corresponding models and their representation. For example, when disassembling the straight cylinder model, the cylinder edges are related to the sides of the rectangle in the flat pattern, and to the length of the circumferences of the bases.

When comparing a model with its flat pattern, problems arise such as the following: To which vertex of the model corresponds a given vertex of its flat pattern? Observing the flat pattern of a cube, can we know the number of faces? Observing at the flat pattern of a solid, can we know the number of faces? How many cuts do we need to make to a model to obtain the flat pattern? Which sides of the flat pattern form an edge in the model?

In order to work on the establishment of relations among the different representations the teacher compares the model properties maintained and the properties that "are broken" in each of them. For example, in a perspective representation of a cylinder, the property of bases being circles is "broken", or in a perspective representation of the cube, the property of all edges being equal and all angles being equal is "broken", property that does show on the corresponding flat patterns.

7. Numerical relations. These types of relationships are studied in several contexts. For example, when finding the numerical characteristics of the prisms, we obtain certain relations such as: the number of edges of a n-agonal prism is equal to 4 times the number of lateral faces plus 2 times the number of sides of the polygon of the base; for regular polyhedrons: the number of edges (sides of polygons of the faces) is equal to number of polygons of the sides of faces multiplied by the number of faces and divided into two.

CONCLUSIONS

In Gonzalez et. al. (2006; 2008) we already pointed out that solids constitute a very important context for the development of mathematical activity and we have presented some features that characterize a competent conduct to teach solid geometry in primary school. These results complement those deduced from observations that we will refer to in the following paragraphs. To introduce the study of geometry in primary school, the competent conduct implies putting into practice the different contents recommended in a training plan for teachers related to the establishment of relationships among geometric contents:

- The use of different contexts with all the possibilities they offer for the establishment of relations among geometric contents of the same and different dimension.

- The establishment of relations among geometric contents of one, tow and three dimensions.

- To emphasize about the multitude of relations among geometric contents. For example, those that arise when considering different solids families and/or their elements: i) cylinders and prisms, cones and pyramids; ii) some polyhedra families (prisms, pyramids); iii) solids families and flat figures, etc; iv) regular polyhedrons and other solids families; v) relations of inscription and duality among regular polyhedrons.

- To work on the transformation of some solid families into others with different objectives, such as: i) focusing attention on seeing them in a more dynamic way; ii) discovering the properties maintained and lost along the transformation; iii) discovering new knowledge; iv) using knowledge that we already have in order to discover new; v) working on the same geometric content in different contexts and times.

- To present the contents regarding the subject knowledge for its teaching without overlooking the contents of the subject itself. For example, to propose different questions with the intention of generating mathematical activity, emphasizing on the relations expressed and paying attention to the type of language used for this purpose; the use of different materials, diagrams and tables with the aim of facilitating the discovery and verbalization with a each time more specific geometric language of the relationships that arise.

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STUDENTS' 3D GEOMETRY THINKING PROFILES

Marios Pittalis, Nicholas Mousoulides & Constantinos Christou

Department of Education, University of Cyprus

This article focuses on the construction, description and testing of a theoretical model for the structure of 3D geometry thinking. We tested the validity and applicability of the model with 269 students (5th to 9th grade) in Cyprus. The results of the study showed that 3D geometry thinking can be described across the following factors: (a) recognition and construction of nets, (b) representation of 3D objects, (c) structuring of 3D arrays of cubes, (d) recognition of 3D shapes' properties, (e) calculation of the volume and the area of solids, and (f) comparison of the properties of 3D shapes. The analysis showed that four different profiles of students can be identified.

INTRODUCTION

Geometry and three-dimensional (3D) thinking is connected to every strand in the mathematics curriculum and to a multitude of situations in real life (Jones & Mooney, 2004, Presmeg 2006). The reasons for including 3D geometry in the school mathematics curriculum are myriad and encompass providing opportunities for learners not only to develop spatial awareness, geometrical intuition and the ability to visualise, but also to develop knowledge and understanding of, and the ability to use, geometrical properties and theorems (Jones, 2002). However, it is widely accepted that the 3D geometry research domain has been neglected and efforts to establish an empirical link between spatial ability and 3D geometry ability have been few in number and generally inconclusive (Presmeg, 2006). Moreover, 3d geometry teaching gets little attention in most mathematics curriculum and students are only engaged in plane representations of solids (Battista 1999; Ben-Haim, Lappan & Houang, 1989). Thus, there is neither a well-accepted theory on 3D geometry learning and teaching, nor a well-substantial knowledge on student's 3D thinking.

The purpose of the present study is twofold. First, it examines the structure of 3D geometry abilities by proposing a model that encompasses most of the previous research in 3D geometry abilities and describes 3D geometry thinking across several dimensions. Second, the study may provide a worthwhile starting point for tracing students' 3D geometry thinking profiles based on empirical data with the purpose of improving instructional practices.

THEORETICAL CONSIDERATIONS

3D Geometry Abilities

For a long time studies on 3D geometry have concentrated mainly on the abilities of students to processes and tasks directly related to school curriculum (NCTM, 2000; Lawrie, Pegg, & Gutierrez, 2000). Following, we describe the main research findings on these 3D geometry abilities.

(a) The ability to represent 3D objects: Plane representations are the most frequent type of representation modes used to represent 3D geometrical objects in school textbooks. However, students have great difficulties in conceptualizing them (Gutierrez, 1992; Ben-Chaim, Lappan, & Houang, 1989). Specifically, students and adults have great difficulties in drawing 3D objects and representing parallel and perpendicular lines in space. Parzysz (1988) pointed out that the representation of a 3D object by means of a 2D figure demands considerable conventionalizing which is not trivial and not learned in school. He concluded that there is a need to explicitly interpret and utilize drawing 3D objects conventions, otherwise, students may misread a drawing and do not understand whether it represents a 2D or a 3D object. **(b) The ability to recognise and construct nets:** Net construction requires students' ability to make translations between 3D objects and 2D nets by focusing and studying the component parts of the objects in both representation modes. Cohen (2003) supported that the visualization of nets involves mental processes that students do not have, but they can develop through appropriate instruction. The transition from the perception of a 3D object to the perception of its net, requires the activation of an appropriate mental act that coordinates the different perspectives of the object. (**c) The ability to structure 3D arrays of cubes:** Tasks related to enumeration of cubes in 3D arrays appear in many school textbooks. For example, images of cuboids composed by unit-sized cubes are used to introduce students to the concept of volume (Ben-Chaim et al., 1989). The development of this ability is not a simple procedure and as a result primary and middle school students fail in these tasks (Battista 1999; Ben-Chaim et al., 1989). Battista (1999) support that students' difficulties to enumerate the cubes that fit in a box can by explained by the lack of the spatial structuring ability and the inability of students to coordinate and integrate to a unified mental model the different views of the structure. **(d) The ability to recognise 3D shapes' properties and compare 3D shapes:** Understanding the properties of a solid equals to understanding how the elements of the solid are interrelated. This understanding may refer to the same object or between objects. The properties of the composing parts, the comparative relations between the same composing parts and the relations between different composing parts compose altogether the properties of a 3D object that students should conceptualize. Although the composing parts of polyhedrons are almost the same, the special characteristics of these parts vary between the different types of polyhedrons (Gutierrez, 1992). **(e) The ability to calculate the volume and the area of solids:** 3D geometry ability is closely connected to students' ability to calculate the volume and surface area of a solid (Owens & Outhred, 2006). Research findings showed that students focus only on the formulas and the numerical operations required to calculate the volume or surface area of a solid and completely ignore the structure of the unit measures (Owens & Outhred, 2006). Based on these findings, researchers affirmed that students should develop two necessary skills to calculate the volume and surface area of a solid: (i) the conceptualization of the numerical operations and the link of the formulas with the structure of the solid, and (ii) the understanding and visualization of the internal structure of the solid.

3D Geometry Levels of Thinking

In plane geometry systematic research efforts have described extensively progressive levels of thinking and define profiles of geometric thinking in various geometric situations. Most of these studies are grounded on Van Hiele's model (Lawrie, Pegg, & Gutierrez, 2000). The van Hiele model of geometric thought outlines the hierarchy of levels through which students progress as they develop of geometric ideas. The model clarifies many of the shortcomings in traditional instruction and offers ways to improve it by focusing on getting students to the appropriate level to be successful in high school Geometry. Gutierrez (1992) extended Van Hiele's model in 3D geometry by analyzing students' behaviour when solving activities of comparing or moving solids is the ground. Students of the first level compare solids on a global perception of the shapes of the solids or some particular elements (faces, edges, vertices) without paying attention to properties such as angle sizes, edge lengths, parallelism, etc. When some one of these mathematical characteristics appears in their answers, it has just a visual role. Students of the second level compare solids based on a global perception of the solids or their elements leading to the examination of differences in isolated mathematical properties (such as angles sizes, parallelism, etc.), apparent from the observation of the solids or known from the solid's name. Their explanations are based on observation. Students of the third level analyze mathematically solids and their elements. Their answers include informal justifications based on isolated mathematical properties of the solids. These properties may be observed in the solids' representations or known from their prior knowledge. Students of the fourth level analyse the solids prior to any manipulation and their reasoning is based on the mathematical structure of the solids or their elements, including properties not seen but formally deduced from definitions or other properties.

THE PURPOSE OF THE STUDY AND THE PROPOSED MODEL

The purpose of the present study is twofold: First, to examine the structure of 3D geometry thinking by validating a theoretical model assuming that 3D geometry thinking consists of the 3D geometry abilities described above. Second, to describe students' 3D geometry thinking profiles by tracing a developmental trend between categories of students. To this end, latent profile analysis, a person-centered analytic strategy, was used to explore students' 3D geometry abilities, allowing for the subsequent description of those patterns in the context of dealing with different forms of 3D geometry situations. In this paper, as it is highlighted in Figure 1, we hypothesized that students' thinking in 3D geometry can be described by six factors that correspond to six distinct 3D geometry abilities. Specifically, the hypothesized model consists of six first order factors which represent the following 3D geometry abilities: (a) Students' ability to recognise and construct nets, i.e., to decide whether a net can be used to construct a solid when folded and to construct nets, (b) students' ability to represent 3D objects, i.e., to draw a 3D object, and to translate from one representational mode to another, (c) students' ability to structure 3D arrays of cubes, i.e., to manipulate 3D arrays of objects, and to enumerate the cubes that fit in a shape, (d) students' ability to recognise 3D shapes' properties, i.e., to identify solids in the environment or in 2D sketches and to realize their structural elements and properties, (e) students' ability to calculate the volume and the area of solids, i.e., to calculate the surface and perceptually estimate the volume of 3D objects without using formulas, and (f) students' ability to compare the properties of 3D shapes.

METHOD

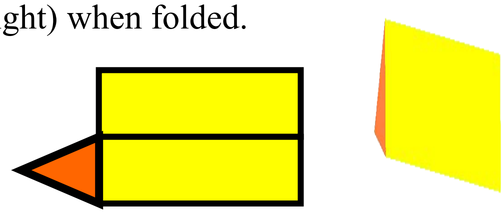
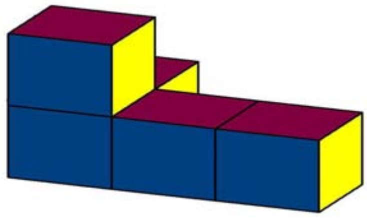
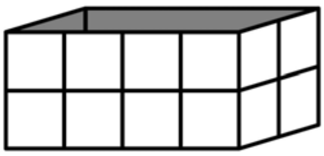
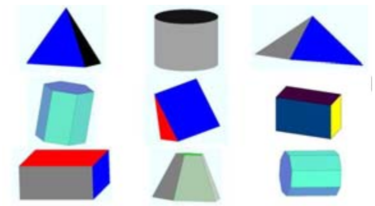
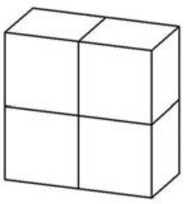
Sample

The sample of this study consisted of 269 students from two primary schools and two middle schools in urban districts in Cyprus. More specifically, the sample consisted of 55 fifth grade students (11 years old), 61 sixth grade students (12 years old), 58 seventh grade students (13 years old), 63 eighth grade students (14 years old) and 42 ninth grade students (15 years old).

Instrument

The 3D geometry thinking test consisted of 27 tasks measuring the six 3D geometry abilities: (a) Four tasks were developed to measure students' ability to recognise and construct nets. Two tasks asked students to recognise the nets of specific solids while the other two asked them to construct or complete the net of specific solids. For example (see Table 1), students had to complete a net in such a manner to construct a triangular prism when folded. (b) Six tasks were developed to capture the nature of the factor "students' ability to represent 3D objects", based on the research conducted by Parzysz (1988) and Ben-Chaim, Lappan, and Houang (1989). Two tasks required students to translate the sketch of a solid from one representational mode to another. For example (see Table 1), students were asked to draw the front, top and side view of an object based on its side projection. (c) Four tasks were used to measure the factor "students' ability to structure 3D arrays of cubes". For example (see Table 1), students were asked to enumerate the cubes that could fit in open and close boxes. (d) Five tasks were developed to measure the factor "students' ability to recognise 3D shapes' properties". For example (see Table 1), students were asked to identify the solids that had minimum eight vertices. The second task asked students to identify the solids that were not cuboids out of twelve objects drawn in a solid form. The other three tasks asked students to enumerate the vertices, edges and faces of three pyramids drawn in transparent view. (e) Four tasks were used as measures of the factor "students' ability to calculate the volume and the area of solids". For example, students were asked to calculate how much wrapping paper is needed to wrap up a cuboid built up by unit-sized cubes. Students should have visualized the object and split its surface area into parts. Two other tasks asked students to calculate the surface area and the volume of cuboids that were presented in a net form (proposed by Battista, 1999). (f) Three tasks were developed to measure the factor "students' ability to compare the properties of 3D shapes”. For example, students were asked to decide whether statements referring to properties of solids were right or wrong (see Table 1). The other two tasks asked students to explore the Euler’s rule and extend it to the case of prisms.

Table 1: Examples of the 3D geometry thinking tasks.

The ability to recognise and construct nets	The ability to represent 3D objects
Complete the following net in a proper manner to construct the triangular prism (at the right) when folded. 	Draw the front, side and top view of the object. 
The ability to structure 3D arrays of cubes	The ability to recognise 3D shapes’ properties
How many unit-sized cubes can fit in the box? 	Circle the solids that have at least 8 vertices. 
The ability to calculate the volume and the area of solids	The ability to compare properties of 3D shapes
Find the area of the box. 	Which of the following statements are wrong? -*The cuboid is not a square prism.* -*The prisms’ and cuboids’ faces are rectangles.* -*The base of the a prism, a cuboid and a pyramid could be a rectangle*

Data Analysis

The structural equation modelling software, MPLUS, was used (Muthen & Muthen, 2007) and three fit indices were computed: The chi-square to its degrees of freedom ratio (x^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA). The observed values for χ^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be lower than .08 to support model fit (Marcoulides & Schumacker, 1996).

RESULTS

In this section, we refer to the main issues of the study. First, we present the results of the analysis, establishing the validity of the latent factors and the viability of the structure of the hypothesized latent factors. Second, we present the exploration of the data for meaningful categories with respect to 3D geometry abilities, and then working up from those categories, we present the characteristics of each 3D geometry thinking profile.

The structure of 3D geometry thinking

In this study, we posited an a-priori (proposed) structure of 3D geometry thinking and tested the ability of a solution based on this structure to fit the data. The proposed model for 3D geometrical thinking consists of six first-order factors. The six first-order factors represent the dimensions of 3D geometry thinking described above: students' ability to recognise and construct nets (F1), students' ability to represent 3D objects (F2), students' ability to structure 3D arrays of cubes (F3), students' ability to recognise 3D shapes' properties (F4), students' ability to calculate the volume and the area of solids (F5), and students' ability to compare the properties of 3D shapes (F6). The six factors were hypothesized to correlate between them (see Figure 1). Figure 1 makes easy the conceptualisation of how the various components of 3D geometry thinking relate to each other.

The descriptive-fit measures indicated support for the hypothesized first order latent factors (CFI=.95, χ^2=375.88, df=301, χ^2/df=1.25, p<0.05, RMSEA=.03). The parameter estimates were reasonable in that all factor loadings were statistically significant and most of them were rather large (see Figure 1). Specifically, the analysis showed that each of the tasks employed in the present study loaded adequately only on one of the six 3D geometry abilities (see the first order factors in Figure 1), indicating that the six factors can represent six distinct functions of students' thinking in 3D geometry. The results of the study showed that the correlations between the six factors are statistically significant and high (see Table 3). The correlation coefficients between F1 with F2 (r=.94, p<.05), F1 with F3 (r=.96, p<.05), F2 with F4 (r=.92, p<.05), F3 with F5 (r=.97, p<.05) and F4 with F6 (r=.92, p<.05) were greater than .90.

Students' 3D Geometry Thinking Profiles

To trace students' different profiles of 3D geometry thinking we examined whether there are different types of students in our sample who could reflect the six 3D geometry abilities. Mixture growth modeling was used to answer this question (Muthen & Muthen, 2007), because it enables specification of models in which one model applies to one subset of the data, and another model applies to another set. The modeling here used a stepwise method-that is, the model was tested under the assumption that there are two, three, and four categories of subjects. The best fitting model with the smallest AIC and BIC indices (see Muthen & Muthen, 2007) was the one involving four categories. Taking into consideration the average class probabilities (not presented due to space limitations), we may conclude that each category has its own characteristics. The means and standard deviations of each of the six 3D geometry abilities across the four categories of students are shown in Table 2, indicating that students in Category 4 outperformed students in Category 3, 2 and Category 1 in all 3D geometry ability factors, students in Category 3 outperformed their counterparts in Categories 2 and 1, while students in Category 2 outperformed their counterparts in Category 1.

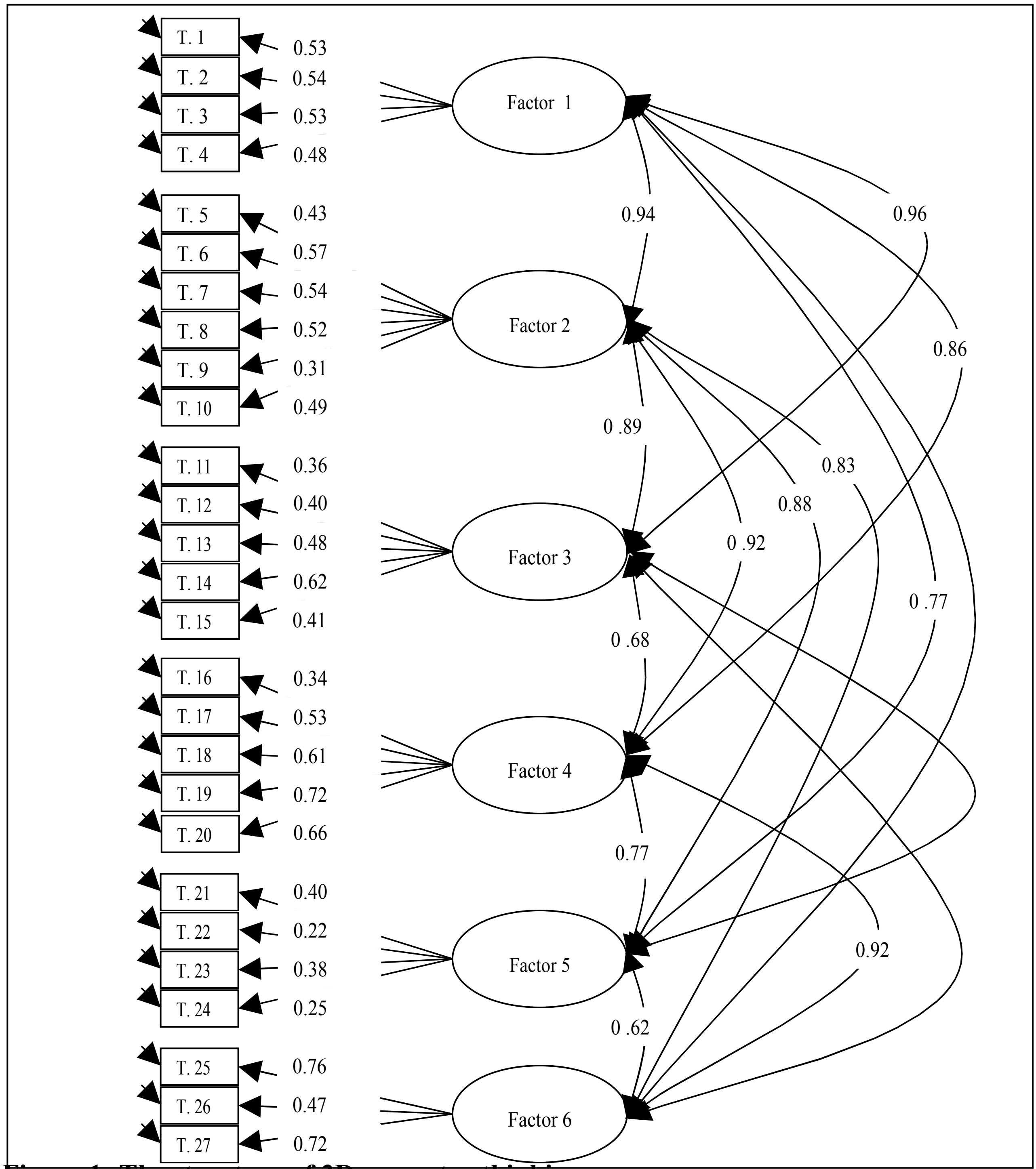


Figure 1: The structure of 3D geometry thinking.

From Table 3, which shows the problems solved by more than 50% or 67% of the students in each category, it can be deduced that there is a developmental trend in students' abilities to complete the assigned tasks of the six factors because success on any problem by more than 67% of the students in a category was associated with such success by more than 67% of the students in all subsequent categories.

Table 2: Means and Standard Deviations of the Four Categories of Students

Category	Factor 1		Factor 2		Factor 3		Factor 4		Factor 5		Factor 6	
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
Categ. 1	0.28	0.16	0.30	0.18	0.15	0.13	0.51	0.17	0.24	0.18	0.10	0.17
Categ. 2	0.54	0.21	0.51	0.17	0.29	0.20	0.71	0.15	0.40	0.21	0.28	0.24
Categ. 3	0.76	0.20	0.71	0.17	0.55	0.22	0.83	0.14	0.49	0.23	0.50	0.26
Categ. 4	0.88	0.14	0.86	0.13	0.83	0.19	0.92	0.07	0.77	0.22	0.78	0.22

The data imply that there are four profiles of students' 3D geometry thinking according to the characteristics of the four categories of students. The first profile of 3D geometry thinking represents the students that recognize in a sufficient way 3D shapes but fail in the other 3D geometry tasks. The second profile of 3D geometry thinking represents the students that do not have any problems in recognizing 3D shapes and have some difficulties in recognizing and constructing nets and representing 3D shapes. Students that belong to the third profile of 3D geometry thinking grasp easily recognizing and representing 3D shapes tasks and recognizing and constructing nets tasks. However, students of the third profile have difficulties in structuring 3D arrays of cubes and comparing 3D shapes' properties. The fourth profile represents the category of students that successfully solves tasks related to the recognition of 3D shapes' properties, the comparison of 3D shapes' properties, the recognition and construction of nets tasks, the structuring of 3D arrays of cubes, the representation of 3D shapes and the calculation of volume and area of solids.

Table 3: Problems Solved by More than 50% or 67% of Students in Each Category

	F1 tasks	F2 tasks	F3 tasks	F4 tasks	F5 tasks	F6 tasks
Category 1				■		
Category 2	■	■		√		
Category 3	√	√	■	√		■
Category 4	√	√	√	√	√	√

■: Problems solved by more than 50%, √: Problems solved by more than 67%

DISCUSSION

The results of the study suggested that 3D geometry thinking can be described across six dimensions based on the following factors which represent six distinct 3D geometry abilities. The first factor is students' ability to recognise and construct nets, by deciding whether a net can be used to construct a solid when folded and by constructing nets. The second factor is students' ability to represent 3D objects, such as drawing a 3D object, constructing a 3D object based on its orthogonal view, and translating from one representational mode to another. The third factor is students' ability to structure 3D arrays of cubes by manipulating 3D arrays of objects, and enumerating the cubes that fit in a shape by spatially structuring the shape. The fourth factor is students' ability to recognise 3D shapes' properties, by identifying solids in the environment or in 2D sketches and realizing their structural elements and properties. The fifth factor is students' ability to calculate the volume and the area of solids. The sixth factor is students' ability to compare the properties of 3D shapes, by comparing the number of vertices, faces and edges, and comparing 3D shapes' properties. The structure of 3D geometry thinking suggests that students need to develop their own 3D geometry skills that integrate the six 3D geometry parameters described above. Based on this assumption, we could also speculate that the most common definition of 3D geometry by other researchers (Gutierrez, 1992) as the knowledge and classification of the various types of solids, in particular polyhedrons, is not sufficient. 3D geometry thinking implies a large variety of 3D geometry situations which do not correspond necessarily to certain school geometry tasks. The results of the study revealed that the six factors are strongly interrelated. The correlation coefficients between the first factor and the second factor, the first factor and the third factor and the third factor and the fifth factor were the stronger ones. This result could be explained by the fact that these factors are strongly related with spatial ability skills.

The second aim concerned the extent to which students in the sample vary according to the tasks provided in the test. The analysis illustrated that four different categories of students can be identified representing four distinct profiles of students. Students of the first profile were able to respond only to the recognition of solids tasks. Students of the second profile were able to recognize and construct nets and represent 3D shapes in a sufficient way. Students of the third profile did not have any difficulties in the recognition and construction of nets and the representation of 3D shapes and furthermore they were able in structuring 3D arrays of cubes and calculating the volume and area of solids in a sufficient way. Students of the fourth profile were able in all the examined tasks.

The identification of students' 3D geometry thinking profiles extended the literature in a way that these four categories of students may represent four developmental levels of thinking in 3D geometry, leading to the conclusion that there are some crucial factors that determine the profile of each student such as the ability to represent 3D objects and the ability to structure 3D arrays of cubes. These two abilities are closely related to spatial visualization skills (Battista, 1991; Parzysz, 1988). This assumption promulgates the call to study in depth the relation of 3D geometry thinking with spatial ability by using a structured quantitative setting.

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