STUDENTS' 3D GEOMETRY THINKING PROFILES

Marios Pittalis, Nicholas Mousoulides & Constantinos Christou

Department of Education, University of Cyprus

This article focuses on the construction, description and testing of a theoretical model for the structure of 3D geometry thinking. We tested the validity and applicability of the model with 269 students (5th to 9th grade) in Cyprus. The results of the study showed that 3D geometry thinking can be described across the following factors: (a) recognition and construction of nets, (b) representation of 3D objects, (c) structuring of 3D arrays of cubes, (d) recognition of 3D shapes' properties, (e) calculation of the volume and the area of solids, and (f) comparison of the properties of 3D shapes. The analysis showed that four different profiles of students can be identified.

INTRODUCTION

Geometry and three-dimensional (3D) thinking is connected to every strand in the mathematics curriculum and to a multitude of situations in real life (Jones & Mooney, 2004, Presmeg 2006). The reasons for including 3D geometry in the school mathematics curriculum are myriad and encompass providing opportunities for learners not only to develop spatial awareness, geometrical intuition and the ability to visualise, but also to develop knowledge and understanding of, and the ability to use, geometrical properties and theorems (Jones, 2002). However, it is widely accepted that the 3D geometry research domain has been neglected and efforts to establish an empirical link between spatial ability and 3D geometry ability have been few in number and generally inconclusive (Presmeg, 2006). Moreover, 3d geometry teaching gets little attention in most mathematics curriculum and students are only engaged in plane representations of solids (Battista 1999; Ben-Haim, Lappan & Houang, 1989). Thus, there is neither a well-accepted theory on 3D geometry learning and teaching, nor a well-substantial knowledge on student's 3D thinking.

The purpose of the present study is twofold. First, it examines the structure of 3D geometry abilities by proposing a model that encompasses most of the previous research in 3D geometry abilities and describes 3D geometry thinking across several dimensions. Second, the study may provide a worthwhile starting point for tracing students' 3D geometry thinking profiles based on empirical data with the purpose of improving instructional practices.

THEORETICAL CONSIDERATIONS

3D Geometry Abilities

For a long time studies on 3D geometry have concentrated mainly on the abilities of students to processes and tasks directly related to school curriculum (NCTM, 2000; Lawrie, Pegg, & Gutierrez, 2000). Following, we describe the main research findings on these 3D geometry abilities.

(a) The ability to represent 3D objects: Plane representations are the most frequent type of representation modes used to represent 3D geometrical objects in school textbooks. However, students have great difficulties in conceptualizing them (Gutierrez, 1992; Ben-Chaim, Lappan, & Houang, 1989). Specifically, students and adults have great difficulties in drawing 3D objects and representing parallel and perpendicular lines in space. Parzysz (1988) pointed out that the representation of a 3D object by means of a 2D figure demands considerable conventionalizing which is not trivial and not learned in school. He concluded that there is a need to explicitly interpret and utilize drawing 3D objects conventions, otherwise, students may misread a drawing and do not understand whether it represents a 2D or a 3D object. **(b) The ability to recognise and construct nets:** Net construction requires students' ability to make translations between 3D objects and 2D nets by focusing and studying the component parts of the objects in both representation modes. Cohen (2003) supported that the visualization of nets involves mental processes that students do not have, but they can develop through appropriate instruction. The transition from the perception of a 3D object to the perception of its net, requires the activation of an appropriate mental act that coordinates the different perspectives of the object. (**c) The ability to structure 3D arrays of cubes:** Tasks related to enumeration of cubes in 3D arrays appear in many school textbooks. For example, images of cuboids composed by unit-sized cubes are used to introduce students to the concept of volume (Ben-Chaim et al., 1989). The development of this ability is not a simple procedure and as a result primary and middle school students fail in these tasks (Battista 1999; Ben-Chaim et al., 1989). Battista (1999) support that students' difficulties to enumerate the cubes that fit in a box can by explained by the lack of the spatial structuring ability and the inability of students to coordinate and integrate to a unified mental model the different views of the structure. **(d) The ability to recognise 3D shapes' properties and compare 3D shapes:** Understanding the properties of a solid equals to understanding how the elements of the solid are interrelated. This understanding may refer to the same object or between objects. The properties of the composing parts, the comparative relations between the same composing parts and the relations between different composing parts compose altogether the properties of a 3D object that students should conceptualize. Although the composing parts of polyhedrons are almost the same, the special characteristics of these parts vary between the different types of polyhedrons (Gutierrez, 1992). **(e) The ability to calculate the volume and the area of solids:** 3D geometry ability is closely connected to students' ability to calculate the volume and surface area of a solid (Owens & Outhred, 2006). Research findings showed that students focus only on the formulas and the numerical operations required to calculate the volume or surface area of a solid and completely ignore the structure of the unit measures (Owens & Outhred, 2006). Based on these findings, researchers affirmed that students should develop two necessary skills to calculate the volume and surface area of a solid: (i) the conceptualization of the numerical operations and the link of the formulas with the structure of the solid, and (ii) the understanding and visualization of the internal structure of the solid.

3D Geometry Levels of Thinking

In plane geometry systematic research efforts have described extensively progressive levels of thinking and define profiles of geometric thinking in various geometric situations. Most of these studies are grounded on Van Hiele's model (Lawrie, Pegg, & Gutierrez, 2000). The van Hiele model of geometric thought outlines the hierarchy of levels through which students progress as they develop of geometric ideas. The model clarifies many of the shortcomings in traditional instruction and offers ways to improve it by focusing on getting students to the appropriate level to be successful in high school Geometry. Gutierrez (1992) extended Van Hiele's model in 3D geometry by analyzing students' behaviour when solving activities of comparing or moving solids is the ground. Students of the first level compare solids on a global perception of the shapes of the solids or some particular elements (faces, edges, vertices) without paying attention to properties such as angle sizes, edge lengths, parallelism, etc. When some one of these mathematical characteristics appears in their answers, it has just a visual role. Students of the second level compare solids based on a global perception of the solids or their elements leading to the examination of differences in isolated mathematical properties (such as angles sizes, parallelism, etc.), apparent from the observation of the solids or known from the solid's name. Their explanations are based on observation. Students of the third level analyze mathematically solids and their elements. Their answers include informal justifications based on isolated mathematical properties of the solids. These properties may be observed in the solids' representations or known from their prior knowledge. Students of the fourth level analyse the solids prior to any manipulation and their reasoning is based on the mathematical structure of the solids or their elements, including properties not seen but formally deduced from definitions or other properties.

THE PURPOSE OF THE STUDY AND THE PROPOSED MODEL

The purpose of the present study is twofold: First, to examine the structure of 3D geometry thinking by validating a theoretical model assuming that 3D geometry thinking consists of the 3D geometry abilities described above. Second, to describe students' 3D geometry thinking profiles by tracing a developmental trend between categories of students. To this end, latent profile analysis, a person-centered analytic strategy, was used to explore students' 3D geometry abilities, allowing for the subsequent description of those patterns in the context of dealing with different forms of 3D geometry situations. In this paper, as it is highlighted in Figure 1, we hypothesized that students' thinking in 3D geometry can be described by six factors that correspond to six distinct 3D geometry abilities. Specifically, the hypothesized model consists of six first order factors which represent the following 3D geometry abilities: (a) Students' ability to recognise and construct nets, i.e., to decide whether a net can be used to construct a solid when folded and to construct nets, (b) students' ability to represent 3D objects, i.e., to draw a 3D object, and to translate from one representational mode to another, (c) students' ability to structure 3D arrays of cubes, i.e., to manipulate 3D arrays of objects, and to enumerate the cubes that fit in a shape, (d) students' ability to recognise 3D shapes' properties, i.e., to identify solids in the environment or in 2D sketches and to realize their structural elements and properties, (e) students' ability to calculate the volume and the area of solids, i.e., to calculate the surface and perceptually estimate the volume of 3D objects without using formulas, and (f) students' ability to compare the properties of 3D shapes.

METHOD

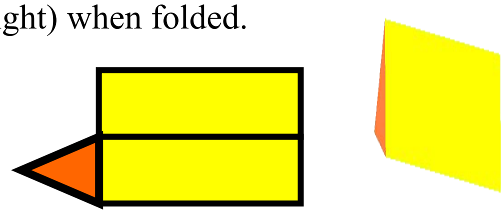
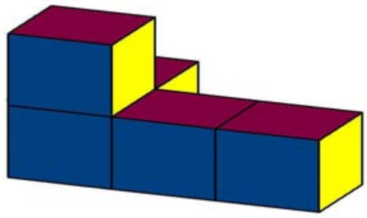
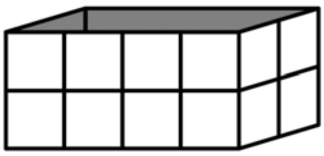
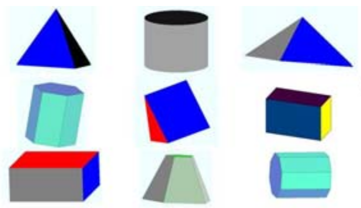
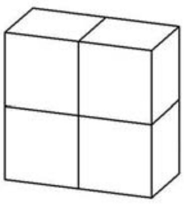
Sample

The sample of this study consisted of 269 students from two primary schools and two middle schools in urban districts in Cyprus. More specifically, the sample consisted of 55 fifth grade students (11 years old), 61 sixth grade students (12 years old), 58 seventh grade students (13 years old), 63 eighth grade students (14 years old) and 42 ninth grade students (15 years old).

Instrument

The 3D geometry thinking test consisted of 27 tasks measuring the six 3D geometry abilities: (a) Four tasks were developed to measure students' ability to recognise and construct nets. Two tasks asked students to recognise the nets of specific solids while the other two asked them to construct or complete the net of specific solids. For example (see Table 1), students had to complete a net in such a manner to construct a triangular prism when folded. (b) Six tasks were developed to capture the nature of the factor "students' ability to represent 3D objects", based on the research conducted by Parzysz (1988) and Ben-Chaim, Lappan, and Houang (1989). Two tasks required students to translate the sketch of a solid from one representational mode to another. For example (see Table 1), students were asked to draw the front, top and side view of an object based on its side projection. (c) Four tasks were used to measure the factor "students' ability to structure 3D arrays of cubes". For example (see Table 1), students were asked to enumerate the cubes that could fit in open and close boxes. (d) Five tasks were developed to measure the factor "students' ability to recognise 3D shapes' properties". For example (see Table 1), students were asked to identify the solids that had minimum eight vertices. The second task asked students to identify the solids that were not cuboids out of twelve objects drawn in a solid form. The other three tasks asked students to enumerate the vertices, edges and faces of three pyramids drawn in transparent view. (e) Four tasks were used as measures of the factor "students' ability to calculate the volume and the area of solids". For example, students were asked to calculate how much wrapping paper is needed to wrap up a cuboid built up by unit-sized cubes. Students should have visualized the object and split its surface area into parts. Two other tasks asked students to calculate the surface area and the volume of cuboids that were presented in a net form (proposed by Battista, 1999). (f) Three tasks were developed to measure the factor "students' ability to compare the properties of 3D shapes”. For example, students were asked to decide whether statements referring to properties of solids were right or wrong (see Table 1). The other two tasks asked students to explore the Euler’s rule and extend it to the case of prisms.

Table 1: Examples of the 3D geometry thinking tasks.

The ability to recognise and construct nets	The ability to represent 3D objects
Complete the following net in a proper manner to construct the triangular prism (at the right) when folded. 	Draw the front, side and top view of the object. 
The ability to structure 3D arrays of cubes	The ability to recognise 3D shapes’ properties
How many unit-sized cubes can fit in the box? 	Circle the solids that have at least 8 vertices. 
The ability to calculate the volume and the area of solids	The ability to compare properties of 3D shapes
Find the area of the box. 	Which of the following statements are wrong? *-The cuboid is not a square prism.* *-The prisms’ and cuboids’ faces are rectangles.* *-The base of the a prism, a cuboid and a pyramid could be a rectangle*

Data Analysis

The structural equation modelling software, MPLUS, was used (Muthen & Muthen, 2007) and three fit indices were computed: The chi-square to its degrees of freedom ratio (x^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA). The observed values for χ^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be lower than .08 to support model fit (Marcoulides & Schumacker, 1996).

RESULTS

In this section, we refer to the main issues of the study. First, we present the results of the analysis, establishing the validity of the latent factors and the viability of the structure of the hypothesized latent factors. Second, we present the exploration of the data for meaningful categories with respect to 3D geometry abilities, and then working up from those categories, we present the characteristics of each 3D geometry thinking profile.

The structure of 3D geometry thinking

In this study, we posited an a-priori (proposed) structure of 3D geometry thinking and tested the ability of a solution based on this structure to fit the data. The proposed model for 3D geometrical thinking consists of six first-order factors. The six first-order factors represent the dimensions of 3D geometry thinking described above: students' ability to recognise and construct nets (F1), students' ability to represent 3D objects (F2), students' ability to structure 3D arrays of cubes (F3), students' ability to recognise 3D shapes' properties (F4), students' ability to calculate the volume and the area of solids (F5), and students' ability to compare the properties of 3D shapes (F6). The six factors were hypothesized to correlate between them (see Figure 1). Figure 1 makes easy the conceptualisation of how the various components of 3D geometry thinking relate to each other.

The descriptive-fit measures indicated support for the hypothesized first order latent factors ($\underline{CFI}$=.95, $\underline{\chi}^2$=375.88, $\underline{df}$=301, $\underline{\chi}^2/\underline{df}$=1.25, $\underline{p}<0.05$, $\underline{RMSEA}$=.03). The parameter estimates were reasonable in that all factor loadings were statistically significant and most of them were rather large (see Figure 1). Specifically, the analysis showed that each of the tasks employed in the present study loaded adequately only on one of the six 3D geometry abilities (see the first order factors in Figure 1), indicating that the six factors can represent six distinct functions of students' thinking in 3D geometry. The results of the study showed that the correlations between the six factors are statistically significant and high (see Table 3). The correlation coefficients between F1 with F2 ($\underline{r}$=.94, $\underline{p}$<.05), F1 with F3 ($\underline{r}$=.96, $\underline{p}$<.05), F2 with F4 ($\underline{r}$=.92, $\underline{p}$<.05), F3 with F5 ($\underline{r}$=.97, $\underline{p}$<.05) and F4 with F6 ($\underline{r}$=.92, $\underline{p}$<.05) were greater than .90.

Students' 3D Geometry Thinking Profiles

To trace students' different profiles of 3D geometry thinking we examined whether there are different types of students in our sample who could reflect the six 3D geometry abilities. Mixture growth modeling was used to answer this question (Muthen & Muthen, 2007), because it enables specification of models in which one model applies to one subset of the data, and another model applies to another set. The modeling here used a stepwise method-that is, the model was tested under the assumption that there are two, three, and four categories of subjects. The best fitting model with the smallest AIC and BIC indices (see Muthen & Muthen, 2007) was the one involving four categories. Taking into consideration the average class probabilities (not presented due to space limitations), we may conclude that each category has its own characteristics. The means and standard deviations of each of the six 3D geometry abilities across the four categories of students are shown in Table 2, indicating that students in Category 4 outperformed students in Category 3, 2 and Category 1 in all 3D geometry ability factors, students in Category 3 outperformed their counterparts in Categories 2 and 1, while students in Category 2 outperformed their counterparts in Category 1.

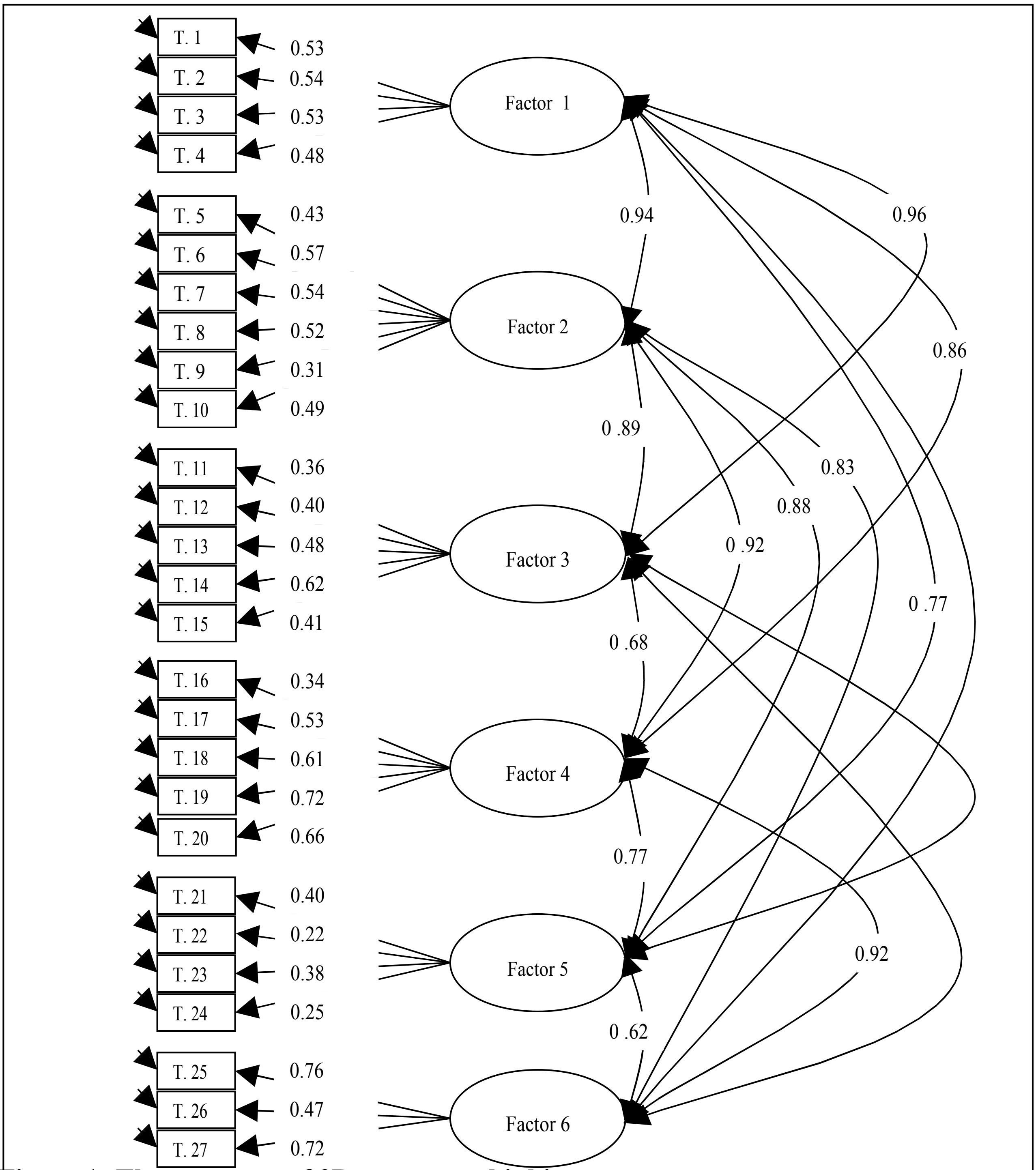


Figure 1: The structure of 3D geometry thinking.

From Table 3, which shows the problems solved by more than 50% or 67% of the students in each category, it can be deduced that there is a developmental trend in students' abilities to complete the assigned tasks of the six factors because success on any problem by more than 67% of the students in a category was associated with such success by more than 67% of the students in all subsequent categories.

Table 2: Means and Standard Deviations of the Four Categories of Students

Category	Factor 1		Factor 2		Factor 3		Factor 4		Factor 5		Factor 6	
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
Categ. 1	0.28	0.16	0.30	0.18	0.15	0.13	0.51	0.17	0.24	0.18	0.10	0.17
Categ. 2	0.54	0.21	0.51	0.17	0.29	0.20	0.71	0.15	0.40	0.21	0.28	0.24
Categ. 3	0.76	0.20	0.71	0.17	0.55	0.22	0.83	0.14	0.49	0.23	0.50	0.26
Categ. 4	0.88	0.14	0.86	0.13	0.83	0.19	0.92	0.07	0.77	0.22	0.78	0.22

The data imply that there are four profiles of students' 3D geometry thinking according to the characteristics of the four categories of students. The first profile of 3D geometry thinking represents the students that recognize in a sufficient way 3D shapes but fail in the other 3D geometry tasks. The second profile of 3D geometry thinking represents the students that do not have any problems in recognizing 3D shapes and have some difficulties in recognizing and constructing nets and representing 3D shapes. Students that belong to the third profile of 3D geometry thinking grasp easily recognizing and representing 3D shapes tasks and recognizing and constructing nets tasks. However, students of the third profile have difficulties in structuring 3D arrays of cubes and comparing 3D shapes' properties. The fourth profile represents the category of students that successfully solves tasks related to the recognition of 3D shapes' properties, the comparison of 3D shapes' properties, the recognition and construction of nets tasks, the structuring of 3D arrays of cubes, the representation of 3D shapes and the calculation of volume and area of solids.

Table 3: Problems Solved by More than 50% or 67% of Students in Each Category

	F1 tasks	F2 tasks	F3 tasks	F4 tasks	F5 tasks	F6 tasks
Category 1				■		
Category 2	■	■		√		
Category 3	√	√	■	√		■
Category 4	√	√	√	√	√	√

■: Problems solved by more than 50%, √: Problems solved by more than 67%

DISCUSSION

The results of the study suggested that 3D geometry thinking can be described across six dimensions based on the following factors which represent six distinct 3D geometry abilities. The first factor is students' ability to recognise and construct nets, by deciding whether a net can be used to construct a solid when folded and by constructing nets. The second factor is students' ability to represent 3D objects, such as drawing a 3D object, constructing a 3D object based on its orthogonal view, and translating from one representational mode to another. The third factor is students' ability to structure 3D arrays of cubes by manipulating 3D arrays of objects, and enumerating the cubes that fit in a shape by spatially structuring the shape. The fourth factor is students' ability to recognise 3D shapes' properties, by identifying solids in the environment or in 2D sketches and realizing their structural elements and properties. The fifth factor is students' ability to calculate the volume and the area of solids. The sixth factor is students' ability to compare the properties of 3D shapes, by comparing the number of vertices, faces and edges, and comparing 3D shapes' properties. The structure of 3D geometry thinking suggests that students need to develop their own 3D geometry skills that integrate the six 3D geometry parameters described above. Based on this assumption, we could also speculate that the most common definition of 3D geometry by other researchers (Gutierrez, 1992) as the knowledge and classification of the various types of solids, in particular polyhedrons, is not sufficient. 3D geometry thinking implies a large variety of 3D geometry situations which do not correspond necessarily to certain school geometry tasks. The results of the study revealed that the six factors are strongly interrelated. The correlation coefficients between the first factor and the second factor, the first factor and the third factor and the third factor and the fifth factor were the stronger ones. This result could be explained by the fact that these factors are strongly related with spatial ability skills.

The second aim concerned the extent to which students in the sample vary according to the tasks provided in the test. The analysis illustrated that four different categories of students can be identified representing four distinct profiles of students. Students of the first profile were able to respond only to the recognition of solids tasks. Students of the second profile were able to recognize and construct nets and represent 3D shapes in a sufficient way. Students of the third profile did not have any difficulties in the recognition and construction of nets and the representation of 3D shapes and furthermore they were able in structuring 3D arrays of cubes and calculating the volume and area of solids in a sufficient way. Students of the fourth profile were able in all the examined tasks.

The identification of students' 3D geometry thinking profiles extended the literature in a way that these four categories of students may represent four developmental levels of thinking in 3D geometry, leading to the conclusion that there are some crucial factors that determine the profile of each student such as the ability to represent 3D objects and the ability to structure 3D arrays of cubes. These two abilities are closely related to spatial visualization skills (Battista, 1991; Parzysz, 1988). This assumption promulgates the call to study in depth the relation of 3D geometry thinking with spatial ability by using a structured quantitative setting.

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