STRENGTHENING STUDENTS' UNDERSTANDING OF 'PROOF' IN GEOMETRY IN LOWER SECONDARY SCHOOL

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This paper reports findings that indicate that as many as 80% of lower secondary age students can continue to consider that experimental verifications are enough to demonstrate that geometrical statements are true - even while, at the same time, understanding that proof is required to demonstrate that geometrical statements are true. Further data show that attending more closely to the matter of the 'Generality of proof' can disturb students' beliefs about experimental verification and make deductive proof meaningful for them.

Key words: Geometrical reasoning, generality of proof, cognitive development, lower secondary school, curriculum design

INTRODUCTION

School geometry is commonly regarded as a key topic within which to teach mathematical argumentation and proof and to develop students' deductive reasoning and creative thinking. Yet while deductive reasoning and proof is central to making progress in mathematics, it remains the case that students at the lower secondary school level have great difficulty in constructing and understanding proof in geometry (Battista, 2007; Mariotti, 2007). Our work focuses on researching, and comparing, the teaching of geometry at the lower secondary school level in countries in the East and in the West, specifically China, Japan and the UK (see, for example, Ding, Fujita, & Jones, 2005; Ding & Jones, 2007; Jones, Fujita & Ding, 2004, 2005). In our research we are interested in students' cognitive needs in the learning of geometrical concepts and thinking, and in principles for classroom practice which would satisfy such needs of students.

In this paper we report selected findings from a series of research projects on the learning and teaching of geometrical proof carried out in Japan where formal proof is intensively taught in the lower secondary school grades (Grades 7-9). We address the issue of students' cognitive needs for conviction and verification and how these needs might be changed and developed through instructional activity. We first present how students in lower secondary schools perceive 'proof' in geometry in terms of the levels of understanding of geometrical proof. We do this by using data collected in 2005 from 418 Japanese students (206 from Grade 8, and 212 from Grade 9). We then offer some suggestions that we have developed from classroom-based research (undertaken since the 1980s) about how we might encourage students' geometrical thinking and understanding of deductive proof in geometry.

Given our data is from studies conducted in Japan, we begin with a short account of the teaching of proof in geometry in Japan.

THE TEACHING OF PROOF IN GEOMETRY IN JAPAN

The specification of the mathematics curriculum for Japan, the 'Course of Study', can be found in the *Mathematics Programme in Japan* (English edition published by the Japanese Society of Mathematics Education, 2000). It should be noted that no differentiation is required in the 'Course of Study', and mixed-attainment classes are common in Japan. 'Geometry' is one of the important topics (the other topics are 'Number and Algebra' and 'Quantitative Relations'). The curriculum states that, in geometry, students must be taught to "understand the significance and methodology of proof" (JSME, 2000, p. 24). In terms of the *Paradigm of Geometry* proposed by Houdement and Kuzniak (Houdement & Kuzniak, 2003), Japanese geometry teaching may be characterized as within the *Geometry II* paradigm (in that axioms are not necessarily explicit and are as close as possible to natural intuition of space as experienced by students in their normal lives).

In terms of Japanese curriculum materials (such as textbooks for Grade 8 and Grade 9 students) our analysis indicates a varying amount of emphasis on 'justifying and proving' (see, for example, Fujita and Jones, 2003; Fujita, Jones and Kunimune, 2008). While the curriculum requires that the principles of how to proceed with mathematical proof are explained in detail, including explanations of 'definitions' and 'mathematical proof', our research repeatedly shows that many students difficulties to understand proof in geometry (for example, Kunimune, 1987; 2000[1]).

In what follows we provide an analytical framework for students' understanding of proof in geometry and then report on our data from three from surveys carried out in 1987, 2000 and 2005.

ASPECTS OF STUDENTS' UNDERSTANDING OF PROOF IN GEOMETRY

In our research, as summarized in this paper, we capture students' understanding of proof in terms of two components: 'Generality of proof' and 'Construction of proof'. The first one these, 'Generality of proof in geometry', recognizes that, on the one hand, students have to understand the generality of proof in geometry, including the universality and generality of geometrical theorems (proved statements), the roles of figures, the difference between formal proof and experimental verification, and so on. The second of these two components, 'Construction of proof in geometry', recognizes that, on the other hand, students also have to learn how to 'construct' deductive arguments in geometry by knowing sufficient about definitions, assumptions, proofs, theorems, logical circularity, and so on.

Considering these two aspects, we work with the following levels of student understanding (we do not have space in this paper to relate these levels to the van Hiele model):

Level I: at this level, students consider experimental verifications are enough to demonstrate that geometrical statements are true. This level is sub-divided into two sub-levels: Level Ia: Do not achieve both ‘Generality of proof’ and ‘Construction of proof’, and Level Ib: Achieved ‘Construction of proof’ but not ‘Generality of proof’

Level II: at this level, students understand that proof is required to demonstrate geometrical statements are true. This level is sub-divided into two sub-levels: Level IIa: Achieved ‘Generality of proof’, but not understand logical circularity, and Level IIb: Understood logical circularity

Level III: at this level, students can understand simple logical chains between theorems

We used the following questions to measure students’ levels of understanding:

Q1 Read the following explanations by three students who demonstrate why the sum of inner angles of triangle is 180 degree.

Student A ‘I measured each angle, and they are 50, 53 and 77. 50+53+77=180. Therefore, the sum is 180 degree.’ Accept/Not accept

Student B ‘I drew a triangle and cut each angle and put them together. They formed a straight line. Therefore, the sum is 180 degree.’ Accept/Not accept

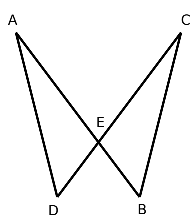
Student C Demonstration by using properties of parallel line (an acceptable proof) Accept/Not accept

Q2 In Figure Q2, prove $AD = CB$ when $\angle A = \angle C$, and $AE=CE$.

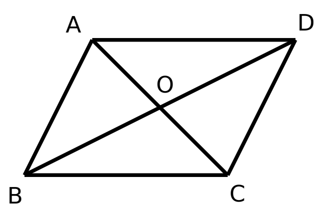
Q3The following argument carefully demonstrates that *the diagonals of a parallelogram intersect at their middle points* (see Figure Q3). ‘In a parallelogram ABCD, let O be the intersection of its diagonals. In Δ ABO and Δ CDO, AB // DC. Therefore, $\angle BAO = \angle DCO$ and $\angle ABO = \angle CDO$. Also, $AB = CD$. Therefore $\Delta ABO \equiv \Delta CDO$. Therefore, $AO = CO$ and $BO = DO$, i.e. the diagonals of a parallelogram intersect at their middle points’

Now, why can we say a) AB // DC, b) $AB = CD$, and c) $\Delta ABO \equiv \Delta CDO$?

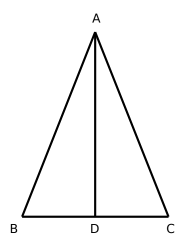
Q4 Do you accept the following argument which demonstrates that *in an isosceles triangle ABC, the base angles are equal*? (see Figure Q4). ‘Draw an angle bisector AD from $\angle A$. In Δ ABD and Δ ACD, $AB = AC$, $\angle BAD = \angle CAD$ and $\angle B = \angle C$. Therefore, $\Delta ABD \equiv \Delta ACD$ and hence $\angle B = \angle C$’. If you do not accept, then write down your reason.



Q2



Q3



Q4

In the above items, Question 1 (Q1) checks whether learners can understand difference between experimental verification and formal proof in geometry. Question 2 (Q2) checks whether learners can understand a simple proof. Q3 checks whether learners can identify assumptions, conclusions and so on in formal proof. Finally, Q4 checks whether learners can identify logical circularity within a formal proof (proof is invalid as '$\angle B = \angle C$' is used to prove '$\angle B = \angle C$'). To achieve Level II, students have to answer Q1 correctly. Students who perform well in Q2 and Q3 can be considered at least at Level Ib as they achieve good understanding in 'Construction of proof'. Figure 1 summarizes the criteria and levels.

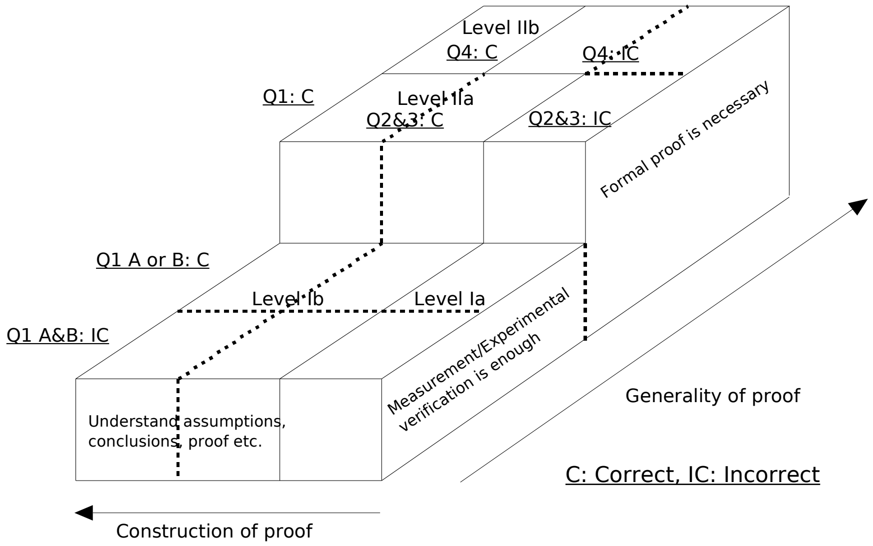


Figure 1: Criteria and levels of generality and proof construction

STUDENTS' UNDERSTANDING OF PROOF IN GEOMETRY

Student surveys were carried out in 1987, 2000 and 2005. One consistent result from these surveys is that over 60% students consider that experimental verification is enough to say it is true that the sum of the inner angles of triangle is 180 degree. Tables 1 and 2 show data collected in 2005 (with 206 students from Grade 8, and 212 students from Grade 9, collected from five different schools).

	Empirical argument using measures (Student A explanation)		**Empirical argument using tearing corners (Student B explanation)**		**Proof (Student C explanation)**	
	Accept	Not accept	Accept	Not accept	Accept	Not accept
Grade 8	62%	32%	70%	21%	74%	15%
Grade 9	36%	58%	52%	38%	80%	6%

Table 1: Results of Q1

The results in Table 1 indicate that, whereas students can accept (or understand) that a formal proof ('Student C' explanation) is a valid way of verification, many also consider experimental verification ('Student A' or 'Student B' explanation) as acceptable. There are, however, changes from Grade 8 to Grade 9, as, by the later grade, more students reject empirical arguments or demonstrations. The likely reason for this is that Grade 9 students have more experience with formal proof, whereas in Grade 8 the students are only just started studying proof (for more on this, see Fujita and Jones, 2003).

Turning now to students' understanding of 'Generality of proof' and 'Construction of proof', the results in Table 2 indicate the following:

- More than half of students can construct a simple proof (Q2).
- Students (in Q3) show relatively good performance for Q3a and Q3b, and these indicate that students have good understanding about deductive arguments of simple properties. Q3c is more difficult as students are required to have knowledge about the conditions of congruent triangles.
- The results of Q4 suggest that more than half of students cannot 'see' why the proof in Q4 is invalid; that is they cannot understand the logical circularity in this proof.

	Q2	Q3a	Q3b	Q3c	Q4
Grade 8	57%	82%	80%	53%	34%
Grade 9	63%	85%	81%	59%	49%

Table 2: Result of Q2-4

In summary, as shown in Table 3, some 90% of Grade 8 and 77% of Grade 9 students were found to be at level I. Data from surveys carried out in 1987 and 2000 show similar results (see Kunimune, 1987, 2000).

Level	**Ia**	**Ib**	**IIa or above**
Grade 8	33%	57%	9%
Grade 9	28%	49%	22%

Table 3: levels of understanding

The result from Grade 9 shows a sight improvement from Grade 8. Using a 2x2 cross-table in which the numbers of level Ia+Ib and IIa or above are considered, the chi-square value is 13.185 ($df=1$, $p<0.01$), and this indicates that the significant improvement can be observed between Grade 8 and Grade 9.

	Level Ia+Ib	**Level IIa or above**
Grade 8	185	19
Grade 9	163	47

Table 4: comparing Grade 8 and Grade 9

MOVING STUDENTS TO DEDUCTIVE THINKING

As evident in a recent review of research on proof and proving by Mariotti (2007, p181), the 'discrepancy' between experimental verifications and deductive reasoning is now a recognized problem. Japan is not an exception to this. Our findings given above indicate that Japanese Grade 8 and 9 students are achieving reasonably well in terms of 'Construction of proof', but not necessarily as well in terms of 'Generality of proof' in geometry. There is a gap between the two aspects. This means that students might be able to 'construct' a formal proof, yet they may not appreciate the significance of such formal proof in geometry. They may believe that a formal proof is a valid argument, while, at the same time, they also believe experimental verification is equally acceptable to 'ensure' universality and generality of geometrical theorems.

Our data for Grade 9 students can be considered as quite concerning, given 80% of students remain at level I in terms of their understanding of proof even after they have studied formal proof at Grade 8 using textbooks where 90% of relevant intended lessons can be devoted for 'justifying and proving' geometrical facts' (Fujita and Jones, 2003). However, we would like to stress that we are still encouraged by the result that 20% of Japanese students achieve relatively sound understanding of proof through everyday mathematics lessons.

Hence, in our research, we turn to the question of working with students on why formal proof is needed. Based on over 10 years of classroom-based research, Kunimune *et al* (2007) propose the following principles for lower secondary school geometry (Grades 7-9) designed to help students appreciate the *need* for formal proofs (in addition to the students being able to construct such proofs):

- Grade 7 lessons to start from problem solving situations such as 'consider how to draw diagonals of a cuboid', and so on; *this develops students' geometrical thinking and provides experiences of mathematical processes that are useful in studying deductive proofs in Grades 8 and 9*;
- Geometrical constructions to be taught in Grade 8; *this replaces the practice of teaching constructions in Grade 7, and then proving these same constructions in Grade 8, as such a gap between the teaching of constructions and their proofs has been found by classroom research to be unhelpful*;
- Grade 8 lessons to provide students with explicit opportunities to examine differences between experimental verifications and deductive proof; *this helps students to appreciate such differences*;
- Grade 8 lessons to start the teaching of the teaching of deductive geometry with a set of already learnt properties which are shared and discussed within the classroom, and used as a form of axioms (a similar idea to that of the 'germ theorems' of Bartolini Bussi, 1996); *this provides students with known starting points for their proofs*.

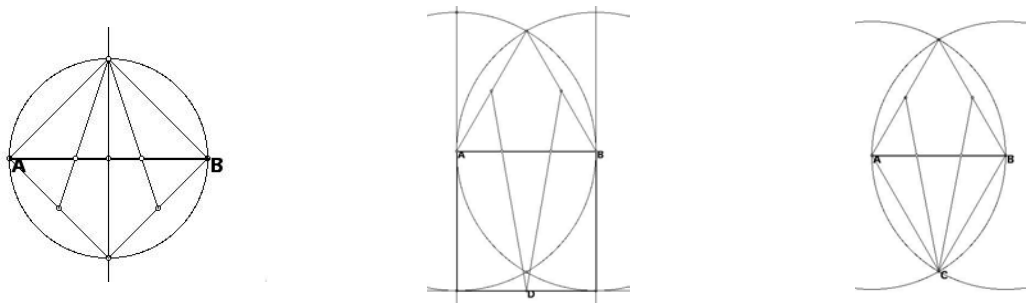
While we do not have space in this paper to provide data to support all these principles, in what follows we substantiate those related to differences between experimental verifications and deductive proof in geometry.

Constructions and proofs

In our experience (Shinba, Sonoda and Kunimune, 2004), while geometrical constructions (with ruler and compasses) can be taught in Grade 7, these constructions are often not proved until Grade 8 (after students have learnt how to prove simple geometrical statements). In a series of teaching experiments, we investigated the use of more complex geometrical constructions (and their proofs) in Grade 8. As an example, one of our lessons in Grade 8 started from the more challenging construction problem 'Let us consider how we can trisect a given straight line AB'.

In our classroom studies, we observed that such lessons are more active for the students. The students could also experience some important processes which bridge between conjecturing and proving. Students could first investigate theorems/properties of geometrical figures through construction activities, and this led them to consider *why* the construction worked. By appropriate instructions by the teachers, the students then started proving the constructions. For example:

Student C: I thought that I could trisect AB when I constructed this (*No. 11* in Figure 2), but I think I found this is not true. So I prove that we cannot trisect the line AB. We just saw the construction No. 8 is true, so I use this approach in my proof. Now, I draw an equilateral triangle on AB (*No. 11'*), and by doing this, we can trisect the AB, and proof is similar to No. 8. Now, compare to this (*No. 11'*) to my construction, and C and D are not in the same place, as the height of the triangle ACB is shorter than the height of the square. We know we can trisect the AB by using this approach, and therefore, my method (*No. 11*) does not work.



No. 8   No. 11   No. 11'

Figure 2: Constructions proposed by students[2]

The data extract above shows that some students in this class start using an already proved statement (i.e. a theorem) to justify why the construction (No. 11 in Figure 2) does not work to trisect the line AB.

Making explicit the differences amongst various argumentations

In a series of lessons for 41 Grade 8 students, tasks were designed and implemented to disturb students' beliefs about experimental verification. In the lessons, students were asked, for example, to compare and discuss various ways of verifying the geometrical statement that the sum of the inner angles of triangles is 180 degrees (this relates to Q1 in the research questionnaire). The angle sum statement was chosen as way of trying to bridge the gap between empirical and deductive approaches, given that students often encounter the angle sum statement in primary schools and they study this again with deductive proof in lower secondary school. While we do not have space in this paper to provide the data from the study, we can provide a summary of ways which can be useful in encouraging students to develop an appreciation of why formal proof is necessary in geometry (for more details, see Kunimune, 1987; 2000).

- Students first exchange their ideas on various ways of verification; they comment on accuracy or generality of experimental verification; they discuss the advantages/disadvantages of experimental verifications.
- Students' comments such as 'A protractor is not always accurate ...', 'It takes time to measure angles, and we cannot see the reason why', 'The triangle is not general', and so on, often cause a state of disequilibrium in students (*viz* Piaget), and make students doubt the universality and generality of experimental verification.
- Students made various comment s on the argument based on 'cutting each angles and fitting them together' (Q1-b). For example, 'I think this is an excellent method as I cannot see any problems in this method', 'This is an easy method to check (whether the sum of inner angles of triangles is 180 degree), 'I think this is a good way, but because we use a piece of paper, I think it can be sometimes inaccurate', and so on.
- Advice from teachers is necessary to encourage students to reflect critically on different ways of verifications (*viz* establishment of 'social norm' in classrooms, Yackel and Cobb, 1996).

Kunimune (1987; 2000) found that, after such lessons, around 40% of students previously at Level Ib have moved to Level II (post-test I). They no longer accept experimental verification and start considering that deductive proof as the only acceptable argument in geometry. A later post-test (post-test II) carried out one month after the lessons found that about 60% of students are at Level IIa. Table 4 (below) summarises the result of the pre and post-tests with five types of cognitive changes observed among students in terms of the levels of understanding of proof in geometry.

An interesting observation is the type *d* in which three students show unexpected behaviour in terms of their cognitive development in that there was a regression from level IIa to Ib. A detailed reason for this is unknown, but, unlike the majority of students, it might be that their states of disequilibrium created rather a 'negative' effect for these students.

In summary, we conclude that the matter of the 'Generality of proof' could usefully be explicitly addressed in geometry lessons in lower secondary schools.

Type	Pre-test	Post-test I	Post-test II	N
a	Level II	Level II	Level II	2
b	Level I	Level II	Level II	13
c	Level I	Level I	Level II	9
d	Level I	Level II	Level I	3
e	Level I	Level I	Level I	14
Level II	2	18	24	

Table 4: Results from Pre- and Post tests

CONCLUDING COMMENTS

This paper outlines research findings from Japan suggesting that, in terms of 'Generality of proof' and 'Construction of proof', many students in lower secondary school remain at Level I where they hold the view that experimental verifications are enough to demonstrate that geometrical statements are true, even after intensive instruction in how to proceed with proofs in geometry. Classroom studies have tested ways of challenging such views about empirical ways of verification which indicate that it is necessary to establish classroom discussions to disturb students' beliefs about experimental verification and to make deductive proof meaningful for them.

NOTES

1. Some papers by Kunimune (1987; 2000) are written in Japanese; this paper, one of outcomes of our collaborative work over five years, contains his main ideas.
2. In No 8 AB is trisected by constructing a square whose diagonal is AB, and joining a vertex and midpoints; In No 11, an equilateral triangle and a square are constructed on AB; In No. 11', AB is trisected by constructing equilateral triangles on AB, and joining a vertex and midpoints.

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