THE GEOMETRICAL REASONING OF PRIMARY AND SECONDARY SCHOOL STUDENTS

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In the present paper comparing the geometrical reasoning of primary and secondary school students was mainly based on the way students confronted and solved specific geometrical tasks: the strategies they used and the common errors appearing in their solutions. This comparison shed light to students' difficulties and phenomena related to the transition from Natural Geometry (the objects of this paradigm of geometry are material objects) to Natural Axiomatic Geometry (definitions and axioms are necessary to create the objects in this paradigm of geometry) and to the inconsistency of the didactical contract implied in primary and secondary school education. These findings stress the need for helping students progressively move from the geometry of observation to the geometry of deduction.

INTRODUCTION

Teaching geometry so that students learn it meaningfully requires an understanding of how students construct their knowledge of various geometric topics (Battista, 1999). This means it is necessary that mathematics educators investigate and mathematics teachers understand how students construct geometrical knowledge as a result of their learning experiences in school. An important aspect of this research direction is the study of the strategies that students use in different geometrical tasks as well as the identification of their mistakes. In the work of Piaget and in the Geneva School we see that errors were for the first time viewed positively, in the sense that they allow the tracing of the reasoning mechanisms adopted by students.

The literature review reveals that the investigation of various issues related to students' geometrical reasoning (knowledge, abilities, strategies, difficulties) is in most cases restricted to the study of groups that come from one educational level. We believe that it is necessary to gather empirical data which would allow the comparison between groups of students in primary and secondary education and would be valuable sources of information regarding aspects of teaching in the two educational levels as well as the difficulties met by students of different age groups.

The transition from elementary to secondary school is recognized as a critical life event, since, progressing from one level of education to the next, students may experience major changes in school climate, educational practices, and social structures (Rice, 2001). Research results reveal substantial agreement that there is often a decline in students' achievement following this transition, but achievement scores tend to recover in the year following the transition (Alspaugh, 1998). In the case of Cyprus, students experience difficulty during the transition from elementary to secondary school which is evident in their performance in most topics, especially in mathematics.

This paper is based upon a research project which investigated the transition from elementary to secondary school geometry in Cyprus, gathering data concerning students' performance in tasks involving two-dimensional geometrical figures, three-dimensional geometrical figures and net-representations of geometrical solids, as well as the students' spatial abilities. In the present paper we focus on the strategies the students used to solve specific geometrical tasks involving two-dimensional figures and we study the kinds of errors that we identified in the students' solutions.

THEORETICAL BACKGROUND

In the present paper we use as explanatory framework Duval's cognitive approach to geometry (Duval, 1995, 1998) and the framework of Geometrical Paradigms proposed by Houdement and Kuzniak (Houdement & Kuzniak, 2003; Houdement, 2007). We also use the concept of the didactical contract, introduced by Brousseau (1984) to interpret some of the students' wrong answers. According to him, the didactical contract is defined as a system of reciprocal expectancies between teacher and pupils, concerning mathematical knowledge. The didactical contract is in large part implicit and is established by the teacher in her teaching practice. The students may interpret the situation put before them and the questions asked to them on the basis of the didactical contract and act accordingly.

A cognitive approach to geometry

Duval (1998) argues that geometry involves three kinds of different cognitive processes – visualization processes, construction processes and reasoning in relation to discursive processes – the synergy of which is necessary for proficiency in geometry. Approaching geometry from a cognitive point of view, he has distinguished four cognitive apprehensions connected to the way a person looks at the drawing of a geometrical figure: perceptual, sequential, discursive and operative (Duval, 1995). Briefly, perceptual apprehension refers to what a person recognizes at first glance when looking at a geometrical figure, while sequential apprehension is required whenever the construction or description of construction of a figure is involved. Discursive apprehension refers to the mathematical properties that cannot be determined through perceptual apprehension of a figure, but must be given through speech or can be derived from the given properties. Operative apprehension depends on the various ways of modifying a given figure. Solving geometrical problems often requires the interactions of these different apprehensions, and "what is called a 'geometrical figure' always associates both discursive and visual representations, even if only one of them can be explicitly highlighted according to the mathematical activity that is required" (Duval, 2006, p.108).

The framework of Geometrical Paradigms

Keeping the idea of 'paradigm' from Kuhn, who used it to explain the development of science, Houdement and Kuzniak (2003) proposed that elementary geometry appears to be split into three various paradigms, characterizing different forms of geometry: Geometry 1 (natural geometry), Geometry 2 (natural axiomatic geometry) and Geometry 3 (formalist axiomatic geometry). The theoretical framework they have developed specifies the nature of the geometrical objects, the use of different techniques and the validation mode accepted in each of the three paradigms. Here we briefly describe the first two geometrical paradigms distinguished by Houdement and Kuzniak (Houdement & Kuzniak, 2003; Houdement, 2007), which mainly concern primary and secondary school students that participated in the present study.

Geometry 1 is intimately related to reality and reasoning is close to experience and intuition. The objects of Geometry 1 are material objects, graphic lines on a paper sheet or virtual lines on a computer screen. Drawing and measurement techniques with ordinary geometrical tools (ruler, set square, compass) as well as experimentation in the sensible world (using techniques such as folding, superposing) are used in this paradigm. New knowledge may be produced based on evidence, experience or reasoning, while a permanent motion between the model and the reality enables the student to 'prove' the assertions.

In Geometry 2 the objects are ideal, so reasoning relies on the mathematical properties of the abstract geometrical objects. A system of definitions and axioms is necessary for the creation of the objects. In this system the axioms are as close as possible to intuition, but making progress and reaching certainty demands demonstrations inside the system. Hypothetical deductive laws are the source of validation.

THE PRESENT STUDY

As noted in the introduction, this paper is based upon a research project which examined primary and secondary school students' geometrical knowledge and abilities related to tasks involving different geometrical figures, as well as their spatial abilities in micro-space. Participants in our study were 1000 primary and secondary school students (488 males and 512 females) from 29 classes of 9 elementary schools and 12 classes of 8 secondary schools in four different districts of Cyprus. Specifically, the sample involved students from three grades (fourth grade – primary school: 332, sixth grade – primary school: 333 and, eighth grade – second grade of secondary school: 335). The mean age of the three grades was as follows: fourth grade, 9.8 years; sixth grade, 11.7 years; eighth grade, 13.9 years. Information concerning the instrument we constructed for the purpose of our research project and the procedure we followed can be found in Panaoura and Gagatsis (2008).

In the present paper we attempt to compare the geometrical reasoning of primary and secondary school students (the three age groups in our study) based on their solutions to three specific geometrical tasks which involved two-dimensional figures (the three tasks are shown in the Appendix). At this point we have to stress that the comparison attempted here does not refer to the levels of success of the three groups of students, since we study students of different age, from different educational levels, with different learning experiences and different cognitive abilities. Using as explanatory framework the theoretical notions presented above, we focus on the strategies and the common errors we identified in students' solutions. In this direction first we present part of the results from our study concerning students' solutions of three geometrical items included in the test and then we discuss these results and students' difficulties under the light of didactic phenomena rising from our research.

RESULTS ON SPECIFIC GEOMETRICAL ITEMS

Item [A]

On the geometrical figure presented in item [A] a square and a right triangle can be identified. In order to give the correct answer, the students had to (a) identify, within the figure presented, the subfigures of the square and the right triangle, (b) pass from 2D to 1D and 'see' that the unknown segment [AC] is one of the square's sides and (c) recall and apply the cognitive unit referring to the property of equal sides in a square. At this point we must note that in the geometry test we included a multiple choice item to examine whether students possess the cognitive unit referring to the property of equal sides in a square. The results presented in Table 1 showed that while a high percentage of the students answered correctly to the specific multiple choice item (61.7% of 4th graders, 85.9% of 6th graders and 86.9% of 8th graders) – indicating they *know* that the four sides of a square are equal – a smaller number of students (especially from primary school) eventually gave a correct answer to the geometrical item [A].

Item	Answer	4th graders	6th graders	8th graders
Multiple choice	Correct	61.7	85.9	86.9
Item [A]	Correct – using properties	36.4	71.8	66.9
	Correct – applying theorem	---	---	18.5
	Wrong – using ruler	8.4	2.1	---
	Wrong – arithmetical operations	6.0	4.8	2.4

Table 1: Students' answers to multiple choice item and item [A] by age group

Crosstabs tables of performance to the multiple choice item by performance to item [A] were obtained for each age group in order to examine what percentage of the students who answered correctly to the specific multiple choice item, did actually solve the geometrical item [A]. The crosstabs results indicated that half of the 4th

grade students and a percentage of 22% of the 6th grade students who gave the correct answer to the multiple choice item (know that the sides of a square are equal) were not able to produce a correct answer to item [A]. The corresponding percentage was 10% in the case of 8th grade students. So it seems that the secondary school students, working in the Natural Axiomatic Geometry paradigm, generally felt the need to use the properties and recalled the right one to solve item [A].

On the other hand, examining at the common errors identified in the students' solutions (Table 1), we notice some primary school students who gave (wrong) answers after using their ruler to measure the unknown segment on the geometrical figure presented on their paper. Additionally, a small number of students of the three age groups tried to combine the arithmetical data of the problem in a random way in arithmetical operations in order to come to an answer.

At this point it is interesting to state that, while the students could give the correct answer to item [A] by simply applying the property of equal sides in a square, we identified 18.5% of the secondary school students who solved the specific geometrical problem by applying Pythagoras' theorem in the subfigure of the right triangle. This performance is probably influenced by a part of the didactical contract according to which they are expected to apply Pythagoras' theorem any time a right triangle is involved in a geometrical figure. On the other hand, the specific performance indicates a difficulty concerning the transition from primary to secondary school. Specifically, the emphasis put on the use of algorithms during mathematics teaching in the secondary school seems to gradually result to the phenomenon that the students feel the safe of using an algorithm to be greater than that of a simple application of a geometrical property.

Items [B] and [C]

In Table 2 we present the results of students' attempts to solve two other geometrical tasks included in our test (item B and item C). Item [B] is a problem given to French students entering middle school (Duval, 2006). Item [C] was constructed for the present study, as an analogous problem to item [B], with two basic differences. First, on the geometrical figure presented in item [B], the subfigures of a circle and a rectangle appear, while on the geometrical figure presented in item [C] the two subfigures identified are a square and a rectangle. Second, the 'visibility' of the geometrical figure (and its subfigures) is less in the case of item [B] due to the specific configuration.

Facing the geometrical problem presented in item [B] a number of students in the present study relied only on a visual perception of the figure (perceptual apprehension) and either considered point E as the middle of [AB] (16.5% of 6th grade students and 9.3% of 8th grade students), or answered that the length of segment [EB] is equal to the circle's ray, "because *it seems to be* equal to the ray" (11.1% of 6th grade students and 9.0% of 8th grade students).

	Item [B]			Item [C]		
Answer	4th graders	6th graders	8th graders	4th graders	6th graders	8th graders
Correct – using properties	15.1	33.3	51.9	46.1	62.2	81.5
Wrong – visual perception (i)	6.6	16.5	9.3	3.3	4.5	0.6
Wrong - visual perception (ii)	8.7	11.1	9.0			
Wrong – using algorithms	10.2	5.4	0.9	11.4	9.9	2.1

Table 2: Students' answers to item [B] and item [C] by age group

In order to solve the item [C], the solver had to identify the two subfigures, to possess and to use the cognitive unit referring to the property of equal sides of a square. As in the case of item [B], a number of students relied only on the visual perception of the given figure and considering point E as the middle of [AB] answered that the length of segment [EB] is equal to 3.5 cm. In both cases perceived features of the geometric figures (relying on a perceptual apprehension of the given figure in each problem) have misled the students as to the mathematical properties involved in the problem solution and have obstructed appreciation of the need for discursive apprehension of the presented geometrical figure.

Finally, it is interesting to note that, as in the case of item [A], there are (mainly primary school) students who tried to give an answer to the items [B] and [C] combining in arithmetical operations the data presented in the geometrical problems. A possible explanation to the specific students' performance is that, according to the implicit didactical contract (Brousseau, 1984) established during the teaching and learning processes in the mathematics classroom – especially the aspect concerning the solution of routine arithmetical word problems – when those students are given a geometrical problem which involves arithmetical data, they suppose that they are expected to combine them in order to give an answer. They probably consider that in this way not only they can give an answer, but they also demonstrate that they have tried to solve the problem by identifying and using the data given in the problem. So, they assume that their teacher will be pleased with their performance!

DISCUSSION

Research about the learning of mathematics and its difficulties "must be based on what students do really by themselves, on their productions, on their voices" (Duval, 2006, p. 104). In this paper we presented some results from our research referring to the solutions of primary and secondary school students in three geometrical items, focusing on the strategies they used and their common errors. Once again we stress that we did not seek to compare students' levels of success, since it is obvious that the students participating in our study have different learning experiences (as far as the amount of experiences and the teaching methods are concerned) and differ in their cognitive development. The comparison of the solutions of the different age groups students shed light to phenomena related to the transition from Natural Geometry to Natural Axiomatic Geometry and to the inconsistency of the didactical contract implied in primary and secondary school education.

The transition from Natural Geometry to Natural Axiomatic Geometry

The passage from Geometry 1 to Geometry 2 is a complex, sensitive and crucial matter (Houdement & Kuzniak, 2003), since these two paradigms are different as far as objects, techniques and validation mode are concerned (Houdement, 2007). Moving from Natural Geometry to Natural Axiomatic Geometry students have to change their theory concerning the nature of the objects and of the space. They are forced to adopt the notion of conceptual objects, the existence of which is based on a definition in an axiomatic system. Consequently, they have to foster new techniques to work relying on the mathematical properties of each abstract geometrical figure.

The findings of the present study indicate that students working in the paradigm of Natural Geometry (mainly primary school students in our study) tend to consider geometrical objects as material objects and specific pictures rather than as theoretical, ideal objects which bear specific properties. This difficulty results to the phenomenon of students trying to solve geometrical problems often relying on the visual perception of the given geometrical figure rather on a mathematical deduction based on the properties of the geometrical objects involved. This phenomenon is related to the students' difficulty to work with geometrical figures as 'figural concepts' (Fischbein, 1993). We call it 'geometrical figure to figural concept' difficulty. As Mariotti (1995) has noted, correct and effective geometrical reasoning is characterized by the interaction and the harmony between figural and conceptual aspects of geometrical entities. In the present study, students working in the Natural Geometry paradigm (mainly primary school students) base their geometrical reasoning on the perceptual apprehension of the geometrical figure presented in a given task and this results to erroneous solutions, since the geometrical properties cannot be determined only through the specific type of apprehension. The perceptual apprehension of a geometrical figure must be under the control of the verbal propositions (discursive apprehension) which are presented in a geometrical problem (Duval, 1998), in such a way that correct geometrical reasoning results through the combination and interaction of the verbal propositions and the geometrical figure. In contrast to the students working under the Natural Geometry paradigm, students working in the Natural Axiomatic Geometry paradigm (mainly amongst secondary school students) focus their efforts on geometrical relations and they confront geometrical tasks based on the properties of geometrical figures (Houdement & Kuzniak, 2003).

Inconsistency of the didactical contract in primary and secondary education

The strategies used by the students in the solution of the presented tasks indicate that the didactical contract which is established among teachers and students concerning geometry learning in primary school education does not discourage all the students from (a) extracting conclusions based on the visual perception of a geometrical figure and (b) giving an answer extracted from random combination of the arithmetical data given in a geometrical problem. These aspects of the didactical contract were not identified to be present in the secondary school education, in the Natural Axiomatic Geometry paradigm, where the emphasis is on the properties of geometrical objects. We call this phenomenon "inconsistency of the didactical contract" among the two education levels concerning the teaching of geometry and further investigation is needed in order to gather information regarding the actual teaching of geometry in primary and secondary schools.

The power of the didactical contract of Natural Axiomatic Geometry

In the case of geometry teaching in the secondary school, the emphasis on learning theorems and continuous practice with close tasks demanding the application of theorems may result in the application of these theorems even in cases that this is not necessary. For example, as a consequence of the continuous practice of the Pythagoras' theorem and the didactical contract formed during teaching, students consider that they are expected to apply Pythagoras' theorem any time a right triangle is involved in a geometrical figure. As we have noted in the results section, attempting to solve a task which could be solved with the mere application of the property of equal sides in a square, almost one fifth of the 8th graders in the present study applied Pythagoras' theorem in the rectangular triangle they identified in the given geometrical figure. The power of the didactical contract in secondary school geometry concerning the application of theorems, leads students to mechanically apply the theorems, especially those that involve an algorithm, feeling safer to use an algorithm than a geometrical property.

Teaching implications and further research

Most of the difficulties that have been identified and discussed in the present study concerning primary and secondary school students' attempts to solve geometrical problems are centred around the issue of the difficulties raised during the transition from Natural Geometry paradigm (where the objects are real, material) to Natural Axiomatic Geometry paradigm (where the objects are conceptual). Subsequently, one of the main goals during the teaching of geometry should be to help students progressively pass from a geometry where objects and their properties are controlled by perception to a geometry where they are controlled by explicitation of properties. But, as Houdement and Kuzniak (2003) note, students and their teachers are not necessarily situated in the same geometrical paradigm, so this is a source of educational misunderstanding. Therefore, we consider essentially important that (prospective) primary and secondary school mathematics teachers are aware of the existence of the different geometrical paradigms (Houdement, 2007) and of the difficulties arising from the fact that plane geometrical figures on paper may be considered by the students in the teaching process during elementary school as if they were real objects (Berthelot & Salin, 1998). Further research is needed in order to prescribe and compare the way mathematics teachers in primary and secondary school approach geometry in their classrooms.

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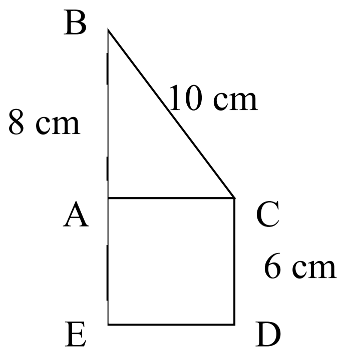
APPENDIX

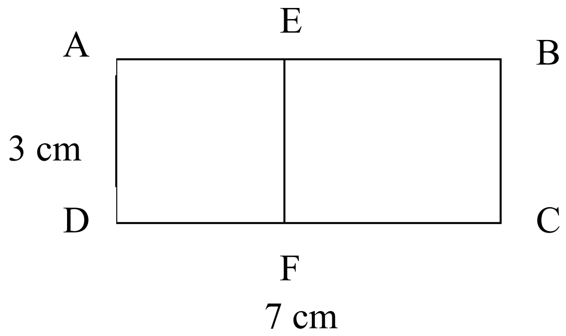
Item A

On the right triangle ABC, BC=10cm and AB=8cm. ACDE is a square (CD=6cm) . Find the length of segment AC.



Item C

On the rectangle ABCD, DC=7cm and AD=3 cm. AEFD is a square. Find the length of segment EB.



Item B

On the figure sketched freehand here (the real lengths are written in cm), are represented a rectangle ABCD and a circle with center A, passing through D.

Find the length of segment EB.