THE EFFECTS OF THE CONCEPT OF SYMMETRY ON LEARNING GEOMETRY AT FRENCH SECONDARY SCHOOL

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This paper relates a part of a bigger research from my Phd (Bulf, 2008) about the symmetry's effects on conceptualization of new mathematical concept. We focus here on the results from students' productions at two different levels at French secondary school, with students who are 12-13 years old and 14-15 y.o. We find out different figural treatments according to the transformation at stake. The results work out the concept of symmetry makes students confused with the transformations of the plan at the beginning of secondary school whereas students seem more familiar with metrical properties relative to the symmetry and develop mathematical reasoning at the end of secondary school.

<u>Key word</u>*: secondary school, geometry, transformations of the plan, symmetry, Geometrical Working Space, conceptualization.*

INTRODUCTION

The constructivist wave suggests that a new knowledge is built from the old one. According to the French curricula (1), the symmetry (reflection through a line) is taught since primary school (through folding and paving), and more deeply during the first year of the secondary school (students are 11-12 years old). Next, the rotational symmetry (reflection through a point) is taught during the second year of the secondary school; the translation is taught during the third year and finally rotation is taught during the last year of the secondary school (students are 14-15 y.o.). One of the specificity of the French curricula is to teach the symmetry as a transformation of the plan even if the term "transformation" is mentioned only at the end of secondary school. Others countries (Italy as for instance) deal with transformations of the plan in the frame of the analytic geometry at high school (students are older than 15 y.o). Then, in this French context, we suppose the concept of symmetry takes part into the learning of the new transformations of the plan. The question is **what are the effects of the symmetry on this learning process?** This paper is the rest of our research, already introduced in CERME 5 (Bulf, 2007).

We do not need to argue that symmetry is part of our "real world" but it is a scientific concept too. Bachelard (1934) points out that "nothing is done, all is building", he adds the notion of obstacles "to set down the problem of scientific knowledge". He describes different kind of obstacles: the obstacle of "the excessive use of familiar images", or the obstacle of "common meaning" and "social representations". Nevertheless, we can not ignore the "real world" may be a help for empirical reasoning. As far as our work is concerned, **we wonder if the concept of symmetry**

may be an "obstacle" or a "help" into the learning process of the new transformations of the plan at secondary school. Several French authors have already pointed out some resistant misunderstandings linked with the concept of symmetry (Grenier & Laborde, 1988) (Grenier, 1990) (Lima, 2006) or linked with the others transformations of the plan, and in particular deal with the dialectic global/punctual (Bkouche, 1992) (Jahn, 1998).

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Our research focuses on the process of conceptualization during the learning of the transformations of the plan. The Vergnaud's theory (Vergnaud, 1991), "the conceptual field theory", analyses the human component of a concept in action. We refer to this framework in order to analyse the students who solve mathematical problem. We focus on the adaptation of the "operational invariants" which are actually defined by the concept-in-action ("relevant or irrelevant notion naturally involved in the mathematics at stake") and theorem-in-action ("proposition assumed right or wrong, used instinctively in the mathematics at stake"). The set of these invariants makes the schemes (notion inspired by Piaget) operate. A scheme is the "invariant organization of behaviour for a class of given situation. The scheme is acting as a whole: it is a functional and dynamical whole, a kind of module finalized by the subject's intention and organized by the way used to reach his goal". The "signifiers" *s* (according to Pressmeg's translation of Saussure's meaning (Presmeg, 2006) is the set of representations of the concept, its properties, and its ways of treatment (language, signs, diagrams, etc.). According to Vergnaud, learning is defined as the adaptation of the schemes from students in a situation of reference.

In order to complete the analysis of students' activities through geometrical problems, we refer to the Houdement and Kuzniak's theoretical framework of Paradigm of Geometry I and Geometry II, and the notion of *Geometrical Working Space* (Houdement & Kuzniak, 2006). Geometry I (GI) is the naive and natural geometry and its validity is the real and sensible world. The deduction operates mainly on material objects through perception and experimentation. Geometry II (GII) is the natural and axiomatic geometry, and its validity operates on an axiomatic system (Euclid). This geometry is modelling reality. The notion of *Geometrical Working Space* (*GWS)* is the study of the environment, organized on a suitable way to articulate these three components: the real and local space, the artefacts (as for instance geometrical tools), and the theoretical references (organized on a model). This *GWS* is used by people who organise it into different aims: the *reference GWS* is seen as the institutional *GWS* from the community of mathematicians, the *idoine GWS* is the efficient one in order to reach a definite goal and the *personal GWS* is the one built with its own knowledge and personal experiments.

Then the main research question is: **How does the concept of symmetry set up the organization and the inferences between the operational invariants relatives to**

the others transformations of the plan into the student's *personal GWS?* And how does this *personal GWS* evolute during secondary school?

METHODOLOGY

We propose a common test to students at two different levels: at the second year, after the teaching of the reflection through a point and, at the fourth year, after the teaching of the rotation. The students are 12-13 y.o. and 14-15 y.o. and have the same mathematics' teacher. We chose the situation of recognition of transformations because it is a usual task all along French secondary school. We define two different tasks from a same configuration with triangles but with different kind of graphical support. These tasks are given to students at two different times. The first task (Fig. 1) suggests a "Global Perception" (we will note GP) because triangles are indicated as a whole with numbers and the transformations are indicated with arrows. This does not mean the students are only involved on a global perception; they may use a punctual perception too. The terms of the problem are: *In each fallow case, indicate which reflection(s), translation(s), rotation(s) transform: a) 1→2 b) 2→3 and c) 1→4. Justify yours answers. If you add marks on the figure, please do not rub out.* The last question *c)* is only given to the students from the last year but we do not analysis the results because we are devoted to the case with reflection(s) and rotation(s). Furthermore, it is only indicated *which reflection(s)* (and not the other transformations) with the students from second year.

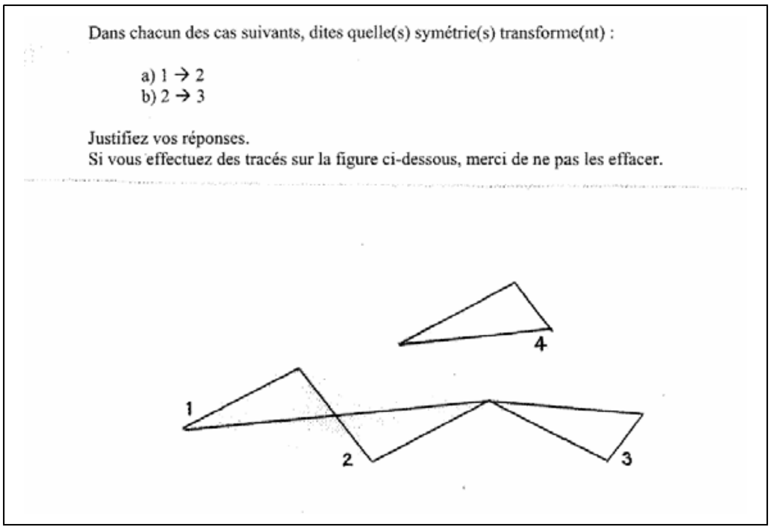
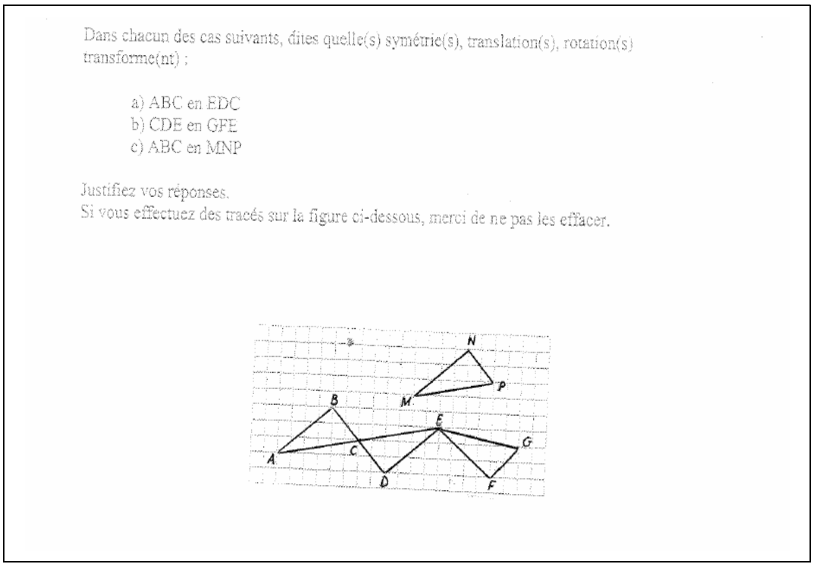


Fig 1: "The triangle situation" in the case called "Global Perception" (GP).

The second task, given one week later, is the same as previously but the terms of the problem suggest a "Punctual Perception" (we will note PP) to the students (Fig. 2). The configuration is given with a squaring and the triangles' tops are called by letters on the pattern and in the terms of the problem (*ABC in EDC*).



Dans chacun des cas suivants, dites quelle(s) symétrie(s), translation(s), rotation(s) transforme(nt) :

a) ABC en EDC
b) CDE en GFE
c) ABC en MNP

Justifiez vos réponses.
Si vous effectuez des tracés sur la figure ci-dessous, merci de ne pas les effacer.

Fig 2: "The triangle situation" in the case called "punctual perception".

These tasks are quite easy for these students (they have to recognize a reflection through a point or a rotation of 180° at the question *a)* and a reflection through an axis at the question *b)*). Different didactical variables are convened and then different students' strategies are implied in both tasks. In particular, the graphical support is different in both case, in the GP one, students' adaptations are wider: they may involve arguments based on superimposition (folding or half-turn) or build strategies based on metrics' arguments (Euclidian Affine Geometry) with measurement or perception. We suppose these latter strategies (with metrical arguments) are more effective in the task PP since there is a squaring and figures are nominated. Mathematical properties are not given as hypothesis in the term of the problems, so different types of metrical properties are acceptable (as for instance "AC=CE" or "AC and CE are almost equals" or even "AC is not equal to CE") but it is assumed a transformation has to be recognized. Moreover, the figural position is actually a didactical variable to consider and we should consider intermediate task (as for instance, without common point, etc.) in order to consolidate the results already got here. However, considering that, we show that students' behaviour changes according to the perception suggested by the task (as expected) but the adaptations imply a different way of figural treatment according to the transformation at stake and according to the students' grade. The aim of this paper is describe the differences between transformations and the influence from the concept of symmetry on these adaptations at these both levels at secondary school.

RESULTS AND DISCUSSION

Student's category according to stability of student's achievement

We collected 29x2=58 productions from students who are 14-15 y.o. and 26x2=52 productions from students who are 12-13 y.o. We classified students' productions according to the stability of their performance on both tasks, i.e. if student proposes a correct answer in the task GP and next if he changes or not his answer in the task called PP. We will write RIGHT (R) or WRONG (W) the student's finale issue on these both tasks. Then, different profiles are exhibited according to the student's achievement at the question a) (the correct transformation is the reflection through a point - or a rotation of 180°) and at the question b) (the correct transformation is a reflection through an axis). Finally, the main student's profiles are presented on the table 3 and table 4, and count at least two students.

Recognition of the reflection through a point (question a)		**Recognition of the reflection through an axis (question b)**		**Number of students**	**Indicative percentage of pupils**
GP	**PP**	**GP**	**PP**		**%**
R	**R**	**R**	**R**	**16**	**≈ 55**
W	**R**	**W**	**R**	**2**	**6,9**
R	**R**	**W**	**W**	**4**	**13,8**
R	**W**	**R**	**R**	**4**	**13,8**
At least one WRONG				**10**	**≈ 34,5**

Tab. 3: Student's profile from the last year of secondary school (14-15 y.o) depending on whether student is successful.

<table>
<tr><th colspan="2">Recognition of the reflection through a point (question a.)</th><th colspan="2">Recognition of the reflection through an axis (question b.)</th><th rowspan="2">Number of students</th><th rowspan="2">Indicative percentage
%</th></tr>
<tr><th>GP</th><th>PP</th><th>GP</th><th>PP</th></tr>
<tr><td>R</td><td>R</td><td>R</td><td>R</td><td>9</td><td>≈ 34,7</td></tr>
<tr><td>R</td><td>R</td><td>W</td><td>W</td><td>3</td><td>11,6</td></tr>
<tr><td>W</td><td>W</td><td>W</td><td>W</td><td>3</td><td>11,6</td></tr>
<tr><td>R</td><td>W</td><td rowspan="2">W</td><td rowspan="2">W</td><td rowspan="2">4</td><td rowspan="2">15,4</td></tr>
<tr><td>W</td><td>R</td></tr>
<tr><td rowspan="2">R</td><td rowspan="2">R</td><td>R</td><td>W</td><td rowspan="2">3</td><td rowspan="2">11,6</td></tr>
<tr><td>W</td><td>R</td></tr>
<tr><td colspan="4">At least one WRONG</td><td>13</td><td>≈ 50</td></tr>
</table>

Tab. 4: Student's profile from the second year of secondary school (12-13 y.o.) depending on whether student is successful.

According to these results, only 34,7 % students from the second year recognize both transformations with successful, whatever the perception suggested by the task; and only 55 % students among students from the last year of secondary school recognize both transformations with successful, whatever the perception suggested by the task.

The students' profiles from the second year are more fragmented than the students'ones from the last year. Therefore, we suppose the student's *Geometrical Working Space (GWS)* from the last year is more stabilized. What we need now is to determine what did each profile (especially what mistakes) and what kind of adaptations they made according to the perception and the transformation at stake.

Local analysis of the *Geometrical Working Space* through the figural treatment according to Duval's meaning

We analyse the *GWS* through its organization between the real space (marks on sheet of paper), the objects of reference from a mathematical model (Euclidian one), and the artefacts (tools, schemes). Inspired by Duval (2005), we focus on the way of treatment of the figure in order to describe these links into the *GWS*. Duval defines different kinds of "figural deconstruction". He opposes "instrumental deconstruction" which implies the use of tools to build the figure and "dimensional deconstruction" which implies links between figural units (for example the points A and B - dimension 0D - indicate the measure AB - dimension 1D) in order to exhibit mathematical properties. The latter deconstruction may imply a mathematical reasoning and suggests a geometrical paradigm closer to GII. Finally, we assume the fact the *GWS* is a favourable environment to analyse the process of conceptualization at stake because, according to Vergnaud's meaning, the notion of representation of the real world is at the heart of the process of conceptualization. Therefore, an analysis of students' productions in term of figural treatment (according to Duval's meaning) is a relevant way to describe the connection between the component of the *GWS* (Object of real world / tools / models of reference) and therefore allows us to approach the process of conceptualization at stake.

Results about students' productions at the end of secondary school (14-15 y.o.)

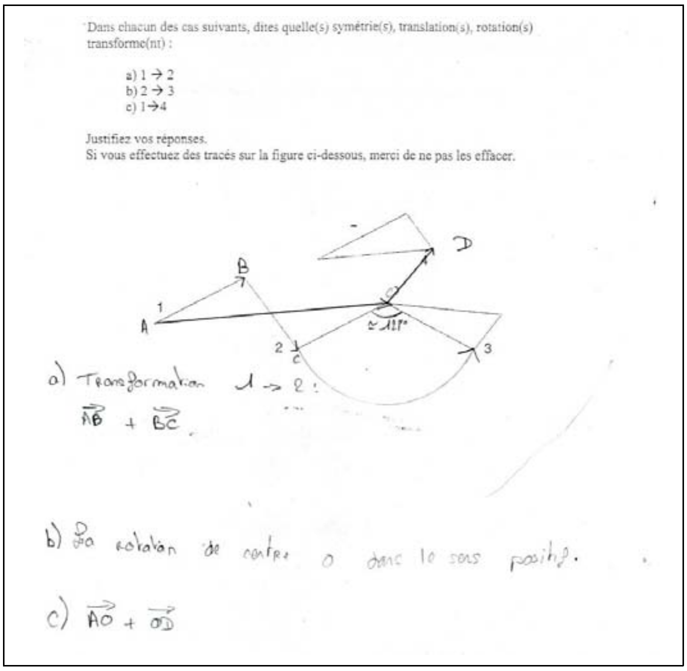
The student's *personal GWS* is adapted to the perception suggested by the task, as expected *a priori*. The operational invariants relative to the recognition of the reflection through an axis are different according to the task. The strategies of superimposition, folding or the use of common references are more present in the case GP than in the case PP.

Students may develop arguments from the Euclidian affine geometry with different kinds of "signifier" (Presmeg, 2006):

- signifier from an "instrumental deconstruction" (Duval, 2005), as for instance the *theorem-in-action of cocyclicity* : pupils use their compasses to test if a couple {point; image} of the figure belong to the same circle and therefore they infer it is a rotation. The language allows the denomination or describes the action.

- signifier from a “dimensional deconstruction” (Duval, 2005) through mathematical symbolism on the drawing (equality of measure, orthogonally, etc.). The language is used to announce the mathematical properties and make deduction.

These adaptations are used not only by students who propose correct answers but with students who propose wrong answers too. At the end of secondary school, we identify only one main kind of mistake made by students in these tasks. Students apply the *theorem-in-action of cocyclicity* at the question b) to recognize a rotation whereas it is actually a reflection through an axis (document 5).



Doc. 5: student’s production with a wrong use of the theorem-in-action of cocyclicity.

We suppose this mistake is from a “cognitive conflict” about the dimension of the mathematical objects at stake with different transformations (between rotation and symmetry). With this theorem-in-action, students do not control the conservation of the measure of the angle with other couples {point; image}. They only refer to an instrumental deconstruction and not to relevant mathematical properties to recognize a rotation. This mistake could be expected if we consider the relative position between triangles (with a common top) but in the case PP, the transformation is given point by point (“CDE in GFE”) and several cases show stronger relation with the figure (because they still use this theorem-in-action) whereas these same students may adapt their strategies according to the task if the recognition of reflection occurs (namely they use a dimensional deconstruction in order to refer to mathematical properties in the case PP). We have already noticed this mistake, called ‘theorem-in-action of cocyclicity” in a pre-test with others students with the same age (Bulf, 2007).

Results about students' productions at the second year of secondary school

If we compare the tab. 3 and tab. 4, students' profiles of 12-13 y.o. are more diversified. The personal *GWS* is still adapted to the perception suggested by the task but not as distinctly as for the students older, i.e. students use references to the real world mainly in the case GP but in the case PP too. On the other hand, they do refer to the Euclidian geometry in the case PP but sometimes in GP too. The mistakes are also more diversified because the adaptations to the perception suggested by the task are different than previously. We distinguish two main sorts of mistake:

- mistakes caused by "contract's effect" in the case PP. The notion of didactical "contract" is designed by Brousseau (1997) as a "relationship […] [which] determines - explicitly to some extent, but mainly implicitly - what each partner, the teacher and the student, will have the responsibility for managing and, in some way or other, be responsible to the other person for managing and, in some way or other, be responsible to the other person for. This system of reciprocal obligation resembles a contract". In our research, students propose mainly exhaustive explanations to solve the task in the case PP. They give too much mathematical properties to justify the transformation. Or, students change their mind and propose "institutional" properties on a wrong way to justify their choice in the case PP whereas their choice in the case GP was correct with naïve arguments from the real word. As for instance, one student justifies correctly the reflection through an axis (question b) in the case GP because he writes "it is possible to fold" but this same student writes, for the same transformation in the case PP, it is a reflection through a point because "in the reflection through a point, the image of a segment is a segment with the same length". This student proposes this same "argument" at the question a) too, but in this case it is coherent. This "institutional" sentence is exactly the same which is given during the classroom. This kind of mistake lets think that the "dimensional deconstruction" (he mentions segments) suggested by students' activity is artificial, and confirm Duval's point of view who pretend this cognitive operation is not self-evident.

- mistakes caused by "amalgam between notion on the same support" according to Artigue's meaning (Artigue, 1990). Students are confused with the reflection through a point and the reflection through an axis, because these both transformations imply the same schemes as for example the global superimposition, cutting in two both sides, the properties of equal distances, etc. In particular, some students recognize a reflection through an axis instead of a reflection through a point in the case GP (question a). Some other students recognize a reflection through a point instead of a reflection through an axis in the task called PP (question b). This kind of amalgam suggests the reflection through an axis is crystallized in a "global perception", at least at the beginning of secondary school.

CONCLUSION AND DISCUSSION

This research is devoted to the analysis of students' productions from two different levels at French secondary school. The students solved the same task given under two different forms (one is called “Global Perception” (GP) and the other one is called “Punctual Perception” (PP)). This research points out that the *personal Geometrical Working Space* is more stabilized for a student at the end of secondary school than for a student at the beginning of secondary school. The schemes of the concept of symmetry are more flexible and can be adapted to the task (arguments can be empirical or from deduction in the frame of Euclidian Affine Geometry according to the perception suggested by the task). These adaptations show a relevant expertise of the dialectic of paradigms GI-GII when the reflection through an axis is involved, for the older students. However, the analyses of the mistakes of these students show a difference of conceptualization between the rotation and symmetry. Rotation involves an “instrumental deconstruction” only, whereas the symmetry may involve “dimensional deconstruction”.

The mistakes made by younger students imply a sort of amalgam between the different symmetries or imply the use of an artificial “dimensional deconstruction”. These mistakes make unstable the *GWS* of these students.

This variation of the use and the effects of the concept of symmetry in the *personal Geometrical Working Space* leave questions about how is managed the concept of symmetry by the teacher during secondary school and how is managed the figural deconstruction. Duval has already mentioned the problem of transmission of the different crossing of figural deconstruction (2D, 1D, 0D) in classroom (Duval, 2005). He points out these different crossings are not so obvious for students, and the difficulty of these crossings are underestimated by teachers and curricula. This point concerns the rest of our research.

NOTES

1. Official instructions: http://eduscol.education.fr/. BO n°10 Hors-Série, 15/10/1998, pp. 106-114 (3e ’s instructions). BO n°5 Hors-Série, 09/09/04, pp. 4-16 (6e ’s instructions). BO n°5 Hors-Série, 25/08/2005, pp. 9-16 (5e ’s instructions).

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