INVESTIGATING COMPARISON BETWEEN SURFACES

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This work is based on a geometrical problem concerning comparison between surfaces, presented to 58 pupils 10-11 years old. We present a worksheet aimed at revealing children's reasoning about visualisation in geometry. We compare the ways in which various problems are tackled by two different groups of students: Group E (experimental) and Group T (traditional). We conclude with some observations about teaching geometry and suggestions for its improvement.

INTRODUCTION

During a lecture to future teachers about fractions, I observed as they were analysing suitable geometric figures, drawn using computer graphics. I realised that these drawings could be useful for investigating geometrical learning. My attention was particularly attracted by different representations of the half of a rectangle. I mentioned my idea to a group of experienced Primary School teachers, and one of them, when she saw figures A, B and C (Figure 1), said: "If the pupils have already worked with fractions, they will certainly use and recognize the concept of half." As in my experience this conclusion is rash and not entirely obvious, I decided to investigate it. Working with the teachers, we prepared a worksheet based on Figures A, B, and C and on a fourth Figure D, expressly created.

The aim of the research is twofold: to investigate the use of the concept of 'half,' and chiefly to study geometrical thinking observing pupils behaviours, with particular reference to registers of representation (Duval, 1998-2006), especially the figural register.

THEORETICAL FRAMEWORK

The concept of *half* and related notations are present in five and six-year-old children (Brizuela, 2006). At this age, children use different semiotic representations (Duval, 1995), but it is difficult for them recognise a half in different representations (Sbaragli, 2008). According to Duval, the passage from a semiotic representation to a different representation is fundamental for a conceptual learning of objects. In particular, he distinguishes two possible kinds of transformation of representation: *conversion* (from a semiotic representation to another, in a different register) and *treatment* (from one semiotic representation to another, in the same register). The half of a geometrical figure is usually presented to children when we introduce fractions, as one of the first examples. Subsequently, teachers move on to writing fractions and to calculating with them, moving from conversions to treatments.

Traditionally in Primary School we use geometrical figures as a suitable tool for teaching and learning geometry. Figures involve a fundamental action for the pupil: looking. The didactical contract (Brousseau, 1986) based on showing requires that

> "the pupil understands what the teacher expects that s/he will see, with the false illusion that both must see the same" (Chamorro, 2006).

If both parties do not see the same, the contract is broken and learning does not take place. So we need to … "teach to see". In geometry, a first problem is created by *perception***,** which may hinder the ways of seeing figures. In other words, the perceptive indicators may be misleading for the qualitative evaluation of the extension of surface or of other magnitudes. Gestalt theory deals with laws of organisation of visual data that lead us to see certain figures rather than others in a picture.

More recent researches show that

> "…it is the task that determines the relation with figures. The way of seeing a figure depends on the activity in which it is involved." (Duval, 2006).

Duval (2006) analyses and classifies the different ways of seeing a figure depending on the geometrical activities presented to pupils. He distinguishes four ways of visualising a figure: by a botanist, a surveyor, a builder or an inventor. Botanists and surveyors have 'iconic visualisation', and perceive the resemblance between a drawing and the shape of an object. Builders and inventors on the other hand have 'non-iconic visualisation', and their perception is based on the *deconstruction of shapes***.** Duval analyses the introduction of supplementary outlines, which he thinks fundamental in 'non-iconic visualisation', in particular he discusses *re-organising outlin*es which allow to reorganise a figure and thus to reveal in it parts and shapes that are not immediately recognizable. .

He also discusses the *méréological decomposition*[1] of shapes, a division of the whole into parts which can be juxtaposed or superimposed, with the aim of reconstructing another figure, often very different to the starting figure. This allows the detection of geometrical properties needed to solve a problem, using an exploration purely visual of the figure initial. He distinguishes three kinds of méréological decomposition: material (with cutting and rebuilding as in a jigsaw puzzle), graphic (using reorganising outlines) and by looking (with the eyes, not "mentally"). We tackled the problem of "which is 'visual' in geometry?" in a research paper (Marchini *et al.*, 2009) where we analysed in-dept the literature on this argument.

In Italian Primary School, *comparison between surfaces* is often reduced to evaluating areas (measurements of extension of surfaces) and to comparing numbers. Teachers tend to determine equivalence of the magnitude of two objects by means of measurement. But "transferring the comparison to the numerical field, we are in fact working with numerical order which doesn't consider the criterion of quantity of

[1] In mathematical logic, mereology is a theory dealing with parts and their respective whole. The term was coined by Łésniewski in 1927, from the Greek word μέρος (*méros*, "part").

magnitude" (Chamorro, 2001). An epistemological slide from geometry to arithmetic occurs. The comparison between surfaces and, in particular, the "equivalence of magnitude" is a fundamental but difficult concept, which requires specific teaching. In previous research we wrote:

> "We did not predict that determining shapes of the same area would be difficult, …. But in fact there were cases where pupils failed to recognise that two congruent rectangles, set at a different way on the sheet of paper, had the same extension." (Marchetti et al., 2005).

The comparison between surfaces is also influenced by the relationship between *shape and surface*: when we present a surface, we present something that has a specific shape. If the shape changes, a younger child might think that the surface changes too. Research shows clearly that pupils under 12 have difficulty in understanding that the shape and the surface of a figure are different (Bang Vinh & Lunzer E., 1965).

RESEARCH METHODOLOGY

We presented the worksheet at the end of the last year of Primary School, to three classes of students 10-11 years old, which had followed two different approaches to geometry. One class had already taken part in an experimental project and the other two classes had received only traditional teaching. We named the first group 'Experimental' (Group E) and the second group 'Traditional' (Group T). Group E consisted of 26 pupils; they had followed a Mathematics Laboratory Project (MLP)[2], during the last three years of Primary School. It focussed on activities that started from a practical problem, such as fencing in a field or tiling a room, and led to the introduction of specific instruments by the teacher as the children perceived the need for them. The early activities involved concrete materials and children using their hands, and geometric instruments and theoretical concepts were introduced in later activities. So Group E did not follow traditional curricular teaching; we presented new activities that were different in terms of both methodology and content. Group T consisted of 32 students from two classes which had followed the traditional mathematics curriculum. Both groups had previously studied and worked with fractions and areas. For Group E, however, the project had opted to present area before perimeter, which is unusual in Italian schools.

Pupils' behaviours were observed as follows: when the teacher presented the worksheet, s/he explained that not was possible to use a rubber, but if necessary children could write their notes and opinions on another sheet of paper. I then analysed the protocols.

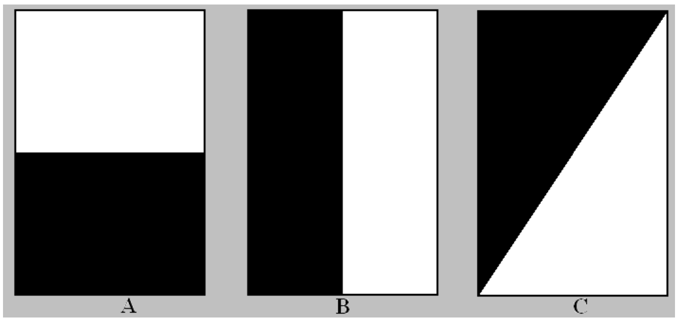
[2] The project was carried out by two researchers, D. Medici and P. Vighi, and two teacher-researchers, P. Marchetti and E. Zaccomer.

THE TASK AND ITS ANALYSIS

In the following pages we present and discuss the worksheet.

A pizza-maker with a lively imagination displays these slices of pizza.

All the slices have one part with only tomato (dark) and one part with only mozzarella (light).



One child wants a slice of pizza with a lot of tomato.

Which slice do you think he or she should choose? Why? ....

Does the slice of pizza below have more mozzarella or more tomato? ....

Why? ....

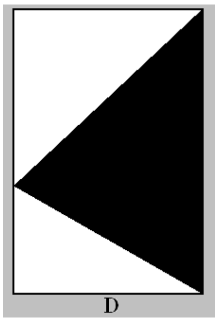


Figure 1: the worksheet

This activity on geometrical figures in the first part lies on the first level of van Hiele's theory, in the final part it lies on the second level, which involves the possibility of seeing inside geometrical figures and seeing and/or making a subdivision into parts (van Hiele, 1986). In the paradigmatic perspective introduced by Houdement and Kusniak (2003), the activity is situated in Geometry I.

Notice that the passage from A to B or C requires 'treatments' inside the register of visual representations. The first question is deliberately ambiguous; the form of the question could lead the child to opt for only one of the slices and, consequently, give a wrong answer. In other words, the question could lead the child to exclude the equivalence of surfaces. The second part of the task presents an unusual geometrical problem. The slice is divided into three parts and the comparison concerns only two quantities of food (two surfaces). There is a different subdivision in half of the same rectangle as before. The question is formulated differently from the first: the problem is the comparison between tomato and mozzarella. Using a supplementary outline helps to find the answer. The main information is in the drawings: rectangles A, B, C and D are congruent (8 cm $\times$ 5.3 cm) and, in particular, in A and B we used the middle point of a side, without specifying this; in other words, we gave implicit data. Figures play an essential role: they are shown against a grey background, with the aim of distinguishing between the whole slice and its parts.

The context of the problem is intended to focus attention on surfaces. The figures in the first part, rectangles and triangles, are familiar; the pupils know the formulas for the calculation of their areas. The last 'slice' is made up of a dark triangle, representing tomato, and two other white triangles, not contiguous, representing just mozzarella. It is an unusual figure which does not appear in textbooks (it may not in fact appear in pizza shops either), but if the sheet of paper is rotated, it probably becomes more familiar as a drawing related to the formula of area of a triangle. For Figure D too, children need to use the concept of half, or they need to "see" congruent parts, or draw supplementary outlines, or calculate areas and verify their equality.

The analysis of A and B by *méréological decomposition* is simpler than for C. In effect there is a difference in the geometry of transformations: in A and B it is sufficient to translate some pieces, while in C rotation is also required. As we saw, D implies cutting the figure and reconstructing congruent parts. We present slice D to investigate pupils' strategies. We want to establish whether children use the same methods for answering both questions, or if D encourages them to try different methods. We also want to observe whether solving the second problem leads pupils to rethink their answers to the first.

RESEARCH RESULTS

The activity is presented in a geometrical context, which often seems to imply the use of specific geometrical tools. In many of the protocols the shift from the geometric register to the numerical register of fractions does not occur: 'conversion' between the registers does not take place.

Only a few answers to the first question (12% in Group E, 6% in Group T) use the concept of "half": "Figures are divided in half", or "Half the space is filled with tomato". The question draws pupils' attention only to the black shapes, or tomato. In other words, children focus on and compare particular parts, rather than looking at the slices globally. It is not by chance that the few answers which are based on "half" make recourse to the relation part-whole (Hart, 1985): "All slices are perfectly divided in the middle and the whole is equal for all figures". Notice that the children use words that are usual in speaking about fractions, not the symbol 1/2. In some cases the concept of half is questionable and 'relative': "I choose pizza C because tomato occupies the "biggest half." The relation shape-surface also emerges: "Even if the pizzas are divided into different shapes, it is still half a slice and the slices are equal".

The "equal extension" of tomato surfaces in A, B, C was recognised by only 6 pupils in Group E and 4 in Group T.

We now analyse different procedures observed for the first part of worksheet.

- *by perception*: children choose slice C because the tomato appears bigger (or "It looks like a piece of pizza") (30% in E and 37% in T). In some cases, the choice is based on exclusion, which may be due to the question: some children verify that A and B have equal quantities of tomato, and they conclude that C must be bigger, without checking. Two pupils choose A because "it is larger," taking account of one dimension only.
- *by subdivision*: here we notice very different behaviours according to the teaching methods adopted. In Group T, only 1 pupil uses méréological decomposition, while in Group E 6 do so. Pupils divide figures B and C by drawing (graphic decomposition) or imagining (decomposition by looking) a continuation of the horizontal line present in slice A which divides the white and black parts. They observe that it is possible to shift some black pieces of B or C in order to obtain A. It is significant that some of them write "If I cut in half …", although they did not see the half in Figures A, B and C.
- *by calculation of area*: only 4 pupils in Group E and 3 in Group T calculate 21.20 cm^2 as measure of three surfaces covered by tomato. There is also a problem of approximation: for figure B, in calculating 5.3 : 2 they stop at the first digit after the decimal point obtaining 2.6 and 2.6×8 make 20.8. Slice B thus seems to have less tomato.
- *by calculation of perimeter*: 6 children in Group E use this method (maybe because perimeter was most recently studied) and 5 in Group T. Their procedures are based on measuring the sides of the black figures and their addition: in this way C appears biggest. This is a manifestation of perimeter-area conflict. (Chamorro, 2002), (Marchetti et al. 2005).
- *by flooring with squares*: based on reproduction of figures on squared paper, often without respect for shapes and measurements, or based on the superimposition of a squared grid, often not regular. Answers are based on counting the number of squares.

In the second part of the worksheet, we recorded 58% correct answers in Group E, and 34% in Group T. Obviously the use of half in the first part of the task is a successful strategy, as it is for the second part.

In Group E, previous methodological decisions and their experience of manipulation led children to tackle the problem in different ways. Some children took scissors, cut the pieces and superimposed two white pieces on the black. They still worked with real and not geometrical objects. Their conclusions may be "They are equal," or not, because there is a problem of approximation: "They differ by a small amount". Recourse to *méréological decomposition* promotes fast and correct answers, based simply on the drawing of a horizontal segment, and the height of the dark triangle. An interesting observation is that a few pupils use the expressions “triangle” or “height of triangle” in their explanations; they write: “I connected the vertex of triangle with the opposite side …” or “I drew a horizontal line …”.

Some pupils make a rough estimate, and make recourse only to perception (26% in Group E, 40 % in Group T). They support their answers as follows: “I can see it,” “The part with tomato is slightly bigger.” In some answers the decision is based on the number of pieces, not on areas: “Mozzarella, because two pieces occupy more space than one.”

Both groups make little use of calculation. One girl wrote: 5.3 × 8 = 42.4 and 42.4 : 2 = 21.2 tomato piece; 5.3 × 5 = 26.5 and 26.5 : 2 = 13.25; 5.3 × 3 = 15.9 and 15.9 : 2 = 7.95; so 13.25 + 7.95 = 21.20 mozzarella piece. This is an example of rigorous application of rules, without geometrical reasoning.

Another boy uses ‘pre-algebraic’ notation and reaches an incorrect conclusion based only on intuition or perception. He tries to explain (Figure 2) that, starting from the area of the rectangle, we can subtract the areas of two white triangles and we obtain the area of the big triangle (black). In the second part, he observes that the sum of the areas of the white triangles is bigger than the area of the ‘big triangle’, but he doesn’t explain why.

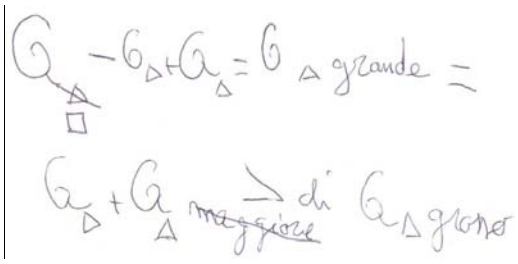


Figure 2: pre-algebraic notation

Some pupils measure two or all sides and multiply them: the idea of multiplication in area calculation is strong, which may be a result of the didactical contract, but there is no understanding of its meaning. We also find mixed procedures: (8 × 5.3) – (8 + 6 + 7) = 42.4–21 = 21.4 area tomato, 42.4–21.4 = 21.0 area mozzarella; the idea is to subtract from the rectangle area the dark triangle area, but the formula for finding the area of a triangle seems not to be known and the pupil calculates the perimeter. Nevertheless one child has a good idea: to obtain the white area as complementary to the black in the rectangle. Only this one boy used this strategy: in fact in school we usually present exercises involving only one shape, and the possibility of calculating an area by subtraction is not introduced.

The solution based on *méréological decomposition* appears the best, and is a successful strategy especially in Group E. We presume that the previous work with Tangram and a different methodological approach helps in the case of Figure D and its parts. Reasoning is based on the use of a supplementary outline (Figure 3).

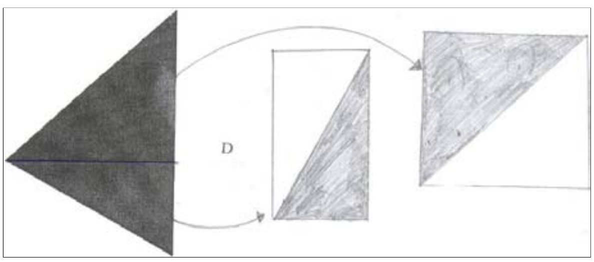


Figure 3: méréological decomposition

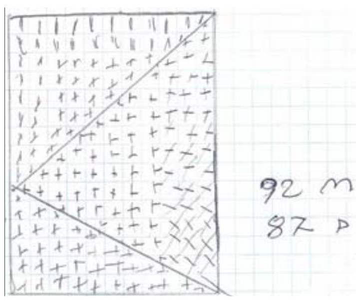


Figure 4: flooring with squares

The idea of measuring with squared paper also appears. In particular, in the protocol reproduced in Figure 4 there is evidence of a lack of understanding: the child counts both squares and pieces of squares and he concludes that the mozzarella area is bigger. In the case of *surface* measurement, schools usually make use of subdivision with squares; there is often no explanation of this method. Moreover it is not suitable for figures with sides that are neither ‘horizontal’ nor ‘vertical’.

Perimeter is used a lot by Group T (18%), but only two pupils use it in Group E (0,07%). It seems that Figure D, which is unusual in traditional teaching, causes the “perimeter-area conflict” and reveals this hidden misconception.

GENERAL CONCLUSIONS

In both groups there were pupils who made no use of geometrical reasoning, but only their eyes. The pizza problem is in fact unusual in that it requires observation of more than one shape and no explicit calculation of its perimeter or area. Often in real life we compare two quantities and we choose the bigger, using common sense rather than mathematics. So one child wrote: “From shapes A, B, and C, I choose C, since it looks like a slice. He was maybe thinking of the shape of a slice of cake. One significant answer came from a child imagining a real pizza, who observed that comparison is impossible, because there is no information about the thickness of the tomato and mozzarella. The analysis of answers confirmed the gap between ‘scholastic’ and ‘real’ problems (Zan, 1998). In other words, the same problem presented in the school or a snack bar may have different solutions. Canapés, in fact, are triangular, obtained by cutting a square along the diagonal, and it could well be that we think we are eating more than if the square of bread were cut in other way.

One week later, the teacher of Group E re-presented the worksheet to her class and encouraged a discussion of pupils’ own solutions. Many quickly recognized the concept of half as a key to the problem and modified their answers. But some children wrote an explanation clearly without conviction. As we wrote previously, in our experience the concept of half does not seem to have been acquired by pupils 10-11 years old. In our opinion, the concept of half needs to be constructed gradually and it is important to work on it with regularity so that it can successfully prepare the ground for introducing fractions.

We also notice that children often use whole numbers as measures of triangle sides: unfortunately in Italy the problem of approximation is neglected. In some cases pupils understand that different numerical results, can be given simply by approximated measurements, but in other cases the children are closely tied to numerical results, even where this conflicts with common sense.

The global analysis of protocols reveals the influence of different teaching methods. Comparison between the protocols of two groups shows clearly the existence of two different behaviours, closely connected to the “social norm” established in classroom (Yackel, Cobb, 1996) according to the “didactical contract”. In Group T, the necessity of following the rules leads to measurement by ruler and the calculation of perimeters and areas. But in Group E, familiarity with manipulation, scissors and so on encourages the use of hands (and the head) (Chamorro, 2008). We observe the presence of an explicit, real geometrical aptitude in Group E, which was probably a result of the MLP. In Group T, traditional geometry and its formulas are prevalent. We surmise that the better results in Group E are closely connected with didactic choices. In other words, the fact that Group E children worked as 'builders' and 'inventors' supports the use of a 'supplementary outline,' which for Duval is fundamental in seeing figures; our experiment confirms his theory of different kinds of visualisation in geometry. Future research will feature an activity based on the same figures but focussing on 'dimensional deconstruction,' defined by Duval as a 'cognitive revolution' for visualisation.

Another important suggestion arises from pupil's approach to the task. Protocol analysis shows that children who use the half or decomposition in shapes A, B and C, use the same concept to investigate D, with the same tools. Vice versa, those who 'found' the half in D, maybe by calculating the area, do not go back to modify their answer to the first part of the task. This points to another critical aspect of traditional teaching, not only in the field of mathematics: exercise books always have be tidy, with no rough work or scribbling, and children are not encouraged to rethink or reflect on work or activities carried out previously. But often sketches and rough drafts can in fact help develop reasoning. We also feel that there should be more encouragement to write up reasoning in the classroom.

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