A THEORETICAL MODEL OF STUDENTS' GEOMETRICAL FIGURE UNDERSTANDING

Eleni Deliyianni*, Iliada Elia *, Athanasios Gagatsis*,

Annita Monoyiou* and Areti Panaoura **

*University of Cyprus, **Frederick University

This study investigated the role of various aspects of apprehension, i.e., perceptual, operative and discursive apprehension, in geometrical figure understanding. Data were collected from 1086 primary and secondary school students. Structural equation modelling affirmed the existence of six first-order factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concepts, three second-order factors indicating the differential effects of the various aspects of geometrical figure apprehension and a third-order factor representing the geometrical figure understanding. It also provided support for the invariance of this structure across the two age groups. However, findings revealed differences between primary and secondary school students' performance and in the way they behaved during the solution of the tasks.

INTRODUCTION AND THEORETICAL FRAMEWORK

In geometry three registers are used: the register of natural language, the register of symbolic language and the figurative register. In fact, a figure constitutes the external and iconical representation of a concept or a situation in geometry. It belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws. As a representation, it becomes more economically perceptible compared to the corresponding verbal one because in a figure various relations of an object with other objects are depicted. However, the simultaneous mobilization of multiple relationships makes the distinction between what is given and what is required difficult. At the same time, the visual reinforcement of intuition can be so strong that it may narrow the concept image (Mesquita, 1998). Geometrical figures are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. A geometrical figure is also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). Therefore the double status of external representation in geometry often causes difficulties to students when dealing with geometrical

This report constitutes a part of the medium research project MED19, funded by the University of Cyprus.

problems due to the interactions between concepts and images in geometrical reasoning (e.g. Mesquita, 1998).

Duval (1995, 1999) distinguishes four apprehensions for a “geometrical figure”: perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, *perceptual apprehension* refers to the recognition of a shape in a plane or in depth. In fact, one’s perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. *Sequential apprehension* is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. *Discursive apprehension* is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). However, it is through *operative apprehension* that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refer to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one made the figure larger or narrower, or slant, while the place way refer to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations. These operations constitute a specific figural processing which provides figures with a heuristic function. In a problem of geometry, one or more of these operations can highlight a figural modification that gives an insight to the solution of a problem.

Even though previous research studies investigated extensively the role of external representations in geometry (e.g. Duval, 1998; Kurina, 2003), the cognitive processes underline the four apprehensions for a “geometrical figure” proposed by Duval (1995, 1999) have not empirically verified yet. Keeping in mind the transition problem from one educational level to another universally (Mullins & Irvin, 2000), our main aim was to confirm a three-order theoretical model concerning the primary and secondary school students’ geometrical figure understanding.

HYPOTHESES AND METHOD

In the present paper four hypotheses were examined: (a) Perceptual, discursive and operative apprehension influence primary and secondary students’ geometrical figure understanding, (b) There are similarities between primary and secondary school students in regard with the structure of their geometrical figure understanding, (c)

Differences exist in the geometrical figure understanding performance of primary and secondary school students and (d) Differences exist in the way primary and secondary school students behave during the solution of the perceptual, discursive and operative apprehension tasks. It should be mentioned that the influence of sequential apprehension in geometrical figure understanding is not investigated since the figure construction is not given much emphasis in the Cypriot curriculum.

The study was conducted among 1086 students, aged 10 to 14, of elementary (Grade 5 and 6) and secondary (Grade 7 and 8) schools in Cyprus (250 in Grade 5, 278 in Grade 6, 230 in Grade 7, 328 in Grade 8). The a priori analysis of the test that was constructed in order to examine the hypotheses of this study is the following:

1. The first group of tasks includes task 1 (Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g) and 2 (Pe2a, Pe2b, Pe2c, Pe2d, Pe2e, Pe2f) concerning students' geometrical figure perceptual ability and their recognition ability, respectively.
2. The second group of tasks includes area and perimeter measurement tasks, namely task 3 (Op3), 4 (Op4), 5 (Op5) and 6 (Op6a, Op6b, Op6c). These tasks examine students' operative apprehension of a geometrical figure. The tasks 3, 4 and 5 require a reconfiguration of a given figure, while task 6 demands the place way of modifying two given figures in a new one in order to be solved.
3. The third group of tasks includes the verbal problems 7 (Ve7), 8 (Ve8), 9 (Ve9), 10 (Ve10) and 11 (Ve11) that correspond to discursive figure apprehension. On the one hand, the verbal problems 7 and 8 demand increased perceptual ability of geometrical figure relations and basic geometrical reasoning. On the other hand, tasks 9, 10 and 11 are verbal area and perimeter measurement problems. In verbal problem 9 visualization (e.g. Presmeg, 2007) facilitates its solution process, while in verbal problems 10 and 11 the concept of epistemological obstacles (Brousseau, 1997) may interfere the way of solving them.

Representative samples of the tasks used in the test appear in the Appendix. Right and wrong or no answers to the tasks were scored as 1 and 0, respectively. The results concerning students' answers to the tasks were codified with Pe, Op and Ve corresponding to perceptual, operative and verbal problem tasks, respectively, followed by the number indicating the exercise number.

In order to explore the structure of the various geometrical figure understanding dimensions a third-order confirmatory factor analysis (CFA) model for the total sample was designed and verified. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI and RMSEA. The following values of the three indices are needed to hold true for supporting an adequate fit of the model: x^2/df < 2, CFI > 0.9, RMSEA < 0.06. The a priori model hypothesized that the variables of all the measurements would be explained by a specific number of factors and each item would have a nonzero loading on the factor it was supposed to measure. The model was tested under the constraint that the error variances of some pair of scores associated with the same factor would have to be equal. A multivariate analysis of variance (MANOVA) was also performed to examine if there were statistically significant differences between primary and secondary school students concerning their understanding in the various geometrical figure dimensions. For the analysis of the collected data the similarity statistical method (Lerman, 1981) was conducted using the statistical software C.H.I.C. (Bodin, Coutourier, & Gras, 2000). A similarity diagram of primary and secondary school students' responses at each task or problem of the test was constructed. The similarity diagram allows for the arrangement of the tasks into groups according to the homogeneity by which they were handled by the students.

RESULTS

Confirmatory factor analysis model. Figure 1 presents the results of the elaborated model, which fitted the data reasonably well [$x^2(220) = 436.86$, CFI = 0.99, RMSEA =0.03, 90%, confidence interval for RMSEA 0.026-0.034]. The first, second and third coefficients of each factor stand for the application of the model in the whole sample (Grade 5 to 8), primary (Grade 5 and 6) and secondary (Grade 7 and 8) school students, respectively. The errors of variables are omitted.

The third-order model which is considered appropriate for interpreting geometrical figure understanding, involves six first-order factors, three second-order factors and one third-order factor. The three second-order factors that correspond to the geometrical figure perceptual (PEA), operative (OPA) and discursive (DIA) apprehension, respectively, are regressed on a third-order factor that stands for the geometrical figure understanding (GFU). Therefore, it is suggested that the type of geometric figure apprehension does have an effect on geometrical figure understanding, verifying our first hypothesis. On the second-order factor that stands for perceptual apprehension the first-order factors F1 and F2 are regressed. The first-order factor F1 refers to the perceptual tasks, while the first-order factor F2 to the recognition tasks. Thus, the findings reveal that perceptual and recognition abilities have a differential effect on geometrical figure perceptual apprehension. On the second-order factor that corresponds to operative apprehension the first-order factors F3 and F4 are regressed. The first-order factor F3 consists of the tasks which require a reconfiguration of a given figure, while the tasks demanding the place way of modifying two given figures in a new one in order to be solved constitute the first-order factor F4. Therefore the results indicate that the ways of figure modification have an effect on operative figure understanding. The first-order factors F5 and F6 are regressed on the second-order factor that stands for discursive apprehension, indicating the effect measurement concept exerts on this type of geometric figure apprehension. To be specific, the first-order factor F5 refers to the verbal problems which demand increased perceptual ability of geometrical figure relations and basic geometrical reasoning, while the first-order factor F6 consists of the verbal perimeter and area problems.

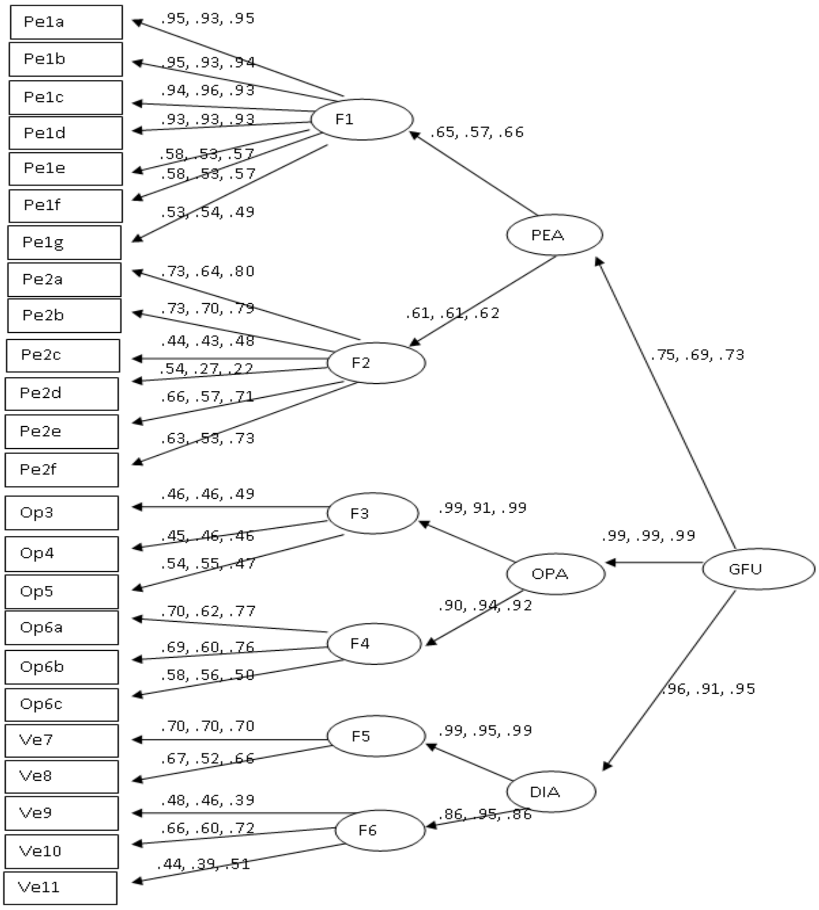


Figure 1. The CFA model of the geometrical figure understanding.

To test for possible similarities between the two educational level groups' geometrical figure understanding, multiple group analysis is applied, where the proposed three-order factor model is validated for elementary and secondary school students separately. The model is tested under the assumption that the relations of the observed variables to the first-order factors, of the six first-order factors to the three second-order factors and of the three second-order factors to the third-order factor will be equal across the two educational levels. The fit indices of the model tested are acceptable [x^2 (485) = 903.78, CFI= 0.97, RMSEA= 0.04, 90% confidence interval for RMSEA= 0.036, 0.044]. Thus, the results are in line with our second hypothesis that the same geometrical figure understanding structure holds for both the elementary and the secondary school students. It is noteworthy that some factor loadings are higher in the group of the secondary school students suggesting that the specific structural organization potency increases across the ages.

The effect of students' educational level. In order to determine whether there are significant differences between primary and secondary school students concerning their performance in the different aspects of geometrical figure understanding, a multivariate analysis of variance (MANOVA) is performed. Table 1 presents the means and the standard deviations for these dimensions in the two educational levels.

Overall, the effect of students' educational level (primary or secondary) is significant (Pillai's $F_{(6,1079)}$=34.43, p<0.001). In particular, the mean value of secondary school students' geometrical figure perceptual ability (F1) is statistically significant higher ($F_{(1,1079)}$=79.51, p<0.001) than the mean value of primary school students. Similarly, the mean value of secondary school students' recognition ability (F2) is statistically significant higher ($F_{(1,1079)}$=38.81, p<0.001) than the mean value of primary school students.

In tasks demanding reconfiguration (F3) secondary school students' performance is statistically significant higher ($F_{(1,1079)}$=74.34, p<0.001) than primary school students' performance. In the same way, the mean value of secondary school students' performance in place way modification tasks (F4) is statistically significant higher in comparison with primary school students' performance ($F_{(1,1079)}$=36.03, p<0.001).

Concerning primary and secondary school students' performance in verbal problems the results are quite different in the two dimensions. Particularly, in verbal problems 7 and 8 (F5) the performance of secondary school students is statistically significant higher ($F_{(1,1079)}$ =105.38, p<0.001) than the performance of primary school students. In contrast, although the performance of secondary school students in verbal problems 9, 10 and 11 (F6) is also higher than the performance of primary school students this difference is not statistically significant ($F_{(1,1079)}$=0.03, p=0.85).

Therefore, the above findings verify the third hypothesis stating that differences exist in the geometrical figure understanding performance of primary and secondary school students. In particular, secondary school students' performance is higher in all the dimensions of the geometrical figure understanding relative to the primary school students' performance.

Level	F1		F2		F3		F4		F5		F6	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
Primary	0.45	0.41	0.62	0.26	0.32	0.31	0.31	0.38	0.38	0.40	0.247	0.28
Secondary	0.66	0.38	0.72	0.27	0.49	0.35	0.45	0.42	0.63	0.40	0.251	0.31

Table 1: Means and standard deviations in geometrical figure apprehension dimensions in primary and secondary school students

Similarity diagrams. Figure 2 and 3 present the similarity diagrams of primary and secondary school students' responses to the tasks of the test. Particularly, in Figure 2 two clusters (i.e., groups of variables) can be distinctively identified. The first cluster consists of the variables corresponding to the perceptual tasks (Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g). In the second cluster the variables representing the recognition, operative and verbal problem solving tasks are included (Pe2a, Pe2c, Pe2b, Pe2e, Ve11, Pe2d, Pe2f, Op6c, Ve7, Ve8, Ve9, Ve10, Op6a, Op6b, Op3, Op5, Op4).

Figure 2. Similarity diagram of primary school students' responses to the test

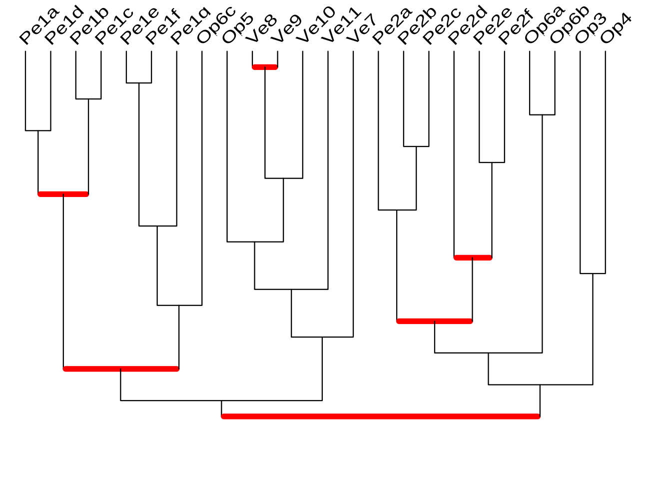


Figure 3. Similarity diagram of secondary school students' responses to the test

In Figure 3, three clusters can be identified. The first cluster includes the perceptual tasks and an operative task (Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g, Op6c). The second cluster consists of an operative task and the verbal problem solving tasks (Op5, Ve8, Ve9, Ve10, Ve11, Ve7). The third cluster involves the recognition tasks and some of the operative tasks (Pe2a, Pe2b, Pe2c, Pe2d, Pe2e, Pe2f, Op6a, Op6b, Op3, Op4). Comparing the two diagrams some relations between the variables remain invariant indicating a stability of the way the primary and secondary school students behave during their solution process (e.g. Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g and Ve8, Ve9, Ve10).

However, differences are observed in many relations of variables. For instance, primary school students behave in a similar way during the solution of the recognition and verbal problem solving tasks, while secondary school students behave in a similar way during the perceptual, some operative and verbal problem solving tasks. Furthermore, in Figure 3 the three clusters are strongly connected with each other indicating that secondary school students behave in a consistent way during the solution of the perceptual, operative and discursive tasks. In contrast, primary school students deal with perceptual tasks in isolation indicating a compartmentalized way of thinking (Duval, 2002). The similarity diagrams' results provide evidence for differences in the way primary and secondary school students behave during the solution of the perceptual, discursive and operative apprehension tasks, verifying the fourth hypothesis.

CONCLUSIONS

This study investigated the role of perceptual, operative and discursive apprehension in geometrical figure understanding. Structural equations modelling affirmed the existence of six first-order factors indicating the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concept, three second-order factors representing perceptual, operative and discursive apprehension and a third-order factor that corresponded to the geometrical figure understanding. It also suggested the invariance of this structure across elementary and secondary school students. Thus, emphasis should be given in all the aspects of geometrical figure apprehension in both educational levels concerning teaching and learning.

Furthermore, differences existed in the geometrical figure understanding performance of primary and secondary school students. Particularly, secondary school students' performance was higher in all the dimensions of the geometrical figure understanding relative to the primary school students' performance. The performance improvement can be attributed to the general cognitive development and learning take place during secondary school. In fact, secondary school curriculum in Cyprus involves many concepts already known and mastered during primary school. This repetition of concepts leads to higher performance even though primary and secondary school instructional practices differ.

Concerning the way students behaved during geometrical tasks solution process it was observed that the behaviour of primary and secondary school students was similar during the solution process of some of the tasks. This finding revealed that geometrical figure understanding stability existed to a certain extent in these students' behaviour. However, in some cases differences were observed in the way the two age groups of students dealt with geometrical figure understanding tasks. To be specific, secondary school students behaved in a consistent way during the solution of the perceptual, operative and discursive tasks. In contrast, primary school students dealt with perceptual tasks in isolation indicating a compartmentalized way of thinking. In fact, the results provided evidence for the existence of three forms of elementary geometry, proposed by Houdement and Kuzniak (2003). We may assume that in this research study, primary school teaching is mainly focused on Geometry I (Natural Geometry) that is closely linked to the perception, is enriched by the experiment and privileges self-evidence and construction. On the other hand, secondary school teaching gives emphasis to Geometry II (Natural Axiomatic Geometry) that it is closely linked to the figures and privileges the knowledge of properties and demonstration. As a result, in the case of primary school students geometrical figure is an object of study and of validation, while in the case of secondary school students geometrical figure supports reasoning and "figural concept" (Fischbein, 1993).

It seems that there is a need for further investigation into the subject with the inclusion of a more extended qualitative and quantitative analysis. In the future an investigation of the way students who master perceptual, operative and discursive apprehension behave in complex activities that require a coordinated approach to these geometrical figure understanding dimensions should be conducted. It would be also interesting to compare the strategies primary and secondary school students use in order to solve perceptual, operative and discursive apprehension tasks. Besides, longitudinal performance investigation in geometrical figure understanding tasks for specific groups of students (e.g. low achievers) as they move from elementary to secondary education should be carried out.

REFERENCES

Bentler, M. P.: 1995, *EQS Structural equations program manual*, Encino, CA, Multivariate Software Inc.

Bodin, A., Coutourier, R., & Gras, R.: 2000, *CHIC : Classification Hiérarchique Implicative et Cohésive-Version sous Windows – CHIC 1.2*, Rennes, Association pour la Recherche en Didactique des Mathématiques.

Brousseau, G.: 1997, *Theory of didactical situations in mathematics*, Dordrecht, The Netherlands, Kluwer.

Duval, R.: 1995, 'Geometrical Pictures: Kinds of Representation and Specific Processes', in R. Sutherland & J. Mason (eds.), *Exploiting mental imagery with computers in mathematical education*, Berlin, Springer, pp. 142- 157.

Duval, R. : 1998, 'Geometry from a cognitive point of view', in C. Mammana & V. Villani (eds.), *Perspectives on the Teaching of Geometry for the 21st century*, Dordrecht, Kluwer Academic, pp. 37-51.

Duval, R.: 1999, *Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for learning*, Retrieved from ERIC ED 466 379.

Duval, R.: 2002, 'The cognitive analysis of problems of comprehension in the learning of mathematics', *Mediterranean Journal for Research in Mathematics Education* 1(2), 1-16.

Fischbein, E.: 1993, 'The theory of figural concepts', *Educational Studies in Mathematics* 24(2), 139-162.

Fischbein, E., & Nachlieli, T.: 1998, 'Concepts and figures in geometrical reasoning', *International Journal of Science Education* 20(10), 1193- 1211.

Houdement, C., & Kuzniak, A.: 2003, 'Elementary geometry split into different geometrical paradigms', in M. Mariotti (ed.), *Proceedings of CERME 3*, Bellaria, Italy, [On line] http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings_draft

Kurina, F.: 2003, 'Geometry - The resource of opportunities', in M. Mariotti (ed.), *Proceedings of CERME 3*, Bellaria, Italy, [On line] http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings_draft

Lerman, I.C.: 1981, *Classification et analyse ordinale des données*, Paris, Dunod.

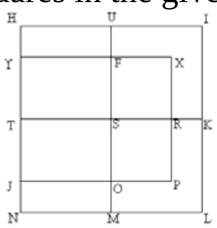
Mesquita, A. L.: 1998, 'On conceptual obstacles linked with external representation in geometry', *Journal of mathematical behavior* 17(2), 183-195.

Mullins, E. R., & Irvin, J. L.: 2000, 'Transition into middle school', *Middle School Journal* 31 (3), 57 -60.

Presmeg, N.: 2007, 'The power and perils of metaphor in making internal connections in trigonometry and geometry', in D. Pitta-Pantazi & G. Philippou (eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education: Working Group 1, The role of images and metaphors in the learning and understanding of mathematics*, Larnaka, Cyprus, ERME, http://www.cyprusisland.com/cerme/, pp. 161-170.

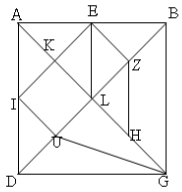
APPENDIX

1. Name the squares in the given figure:



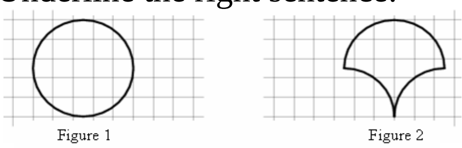
(Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g)

2. Recognize the figures in the parenthesis (KEZL, IEZU, EZHL, IKGU, LGU, BIL)



(Pe2a, Pe2b, Pe2c, Pe2d, Pe2e, Pe2f)

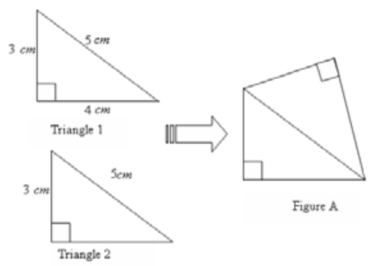
3. Underline the right sentence:



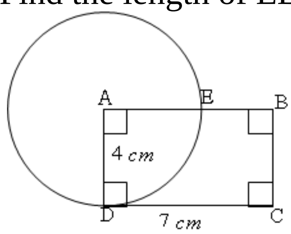
(Op4)

a) Fig. 1 has equal perimeter with Fig. 2
b) Fig. 1 has smaller perimeter than Fig. 2
c) Fig. 1 has bigger perimeter than Fig. 2

4.Peter combines Triangle 1 and Triangle 2 making Figure A. Calculate the perimeter of Figure A. (Op6a)



5. In the following figure the rectangle ABCD and the circle with centre A are given. Find the length of EB.



(Ve7)

6. Themistoklis has a square field with side 40m. He wants to construct a square swimming pool which is far from each side of the field 15m. Find the swimming pool perimeter. (Ve9)