THE NECESSITY OF TWO DIFFERENT TYPES OF APPLICATIONS IN ELEMENTARY GEOMETRY

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This article connects the results of an ontological investigation on elementary geometry to normative questions on educational goals of modelling. The main thesis consists in the assumption that there are two different types of applications in geometry and that they both are necessary and not exchangeable by each other: The first one contains simple applications which are paradigmatic examples to learn basic geometrical concepts; the second one includes more complex ones. It is claimed that a normative discussion on education goals of modelling is only possible as far as the second type is concerned. As a result, the debate on modelling differs in the scope of geometry significantly from similar considerations relative to other parts of mathematics, and that by an ontological and not normative reason.

A CASE STUDY TO RETHINK THE ROLE OF APPLICATIONS

This article is a result of a qualitative study concerning teachers' beliefs (Calderhead 1996) about teaching geometry at German higher level secondary schools (the so-called Gymnasien) including goals, contents, methods and connections to the teachers' broader understanding of mathematics as a whole system. The theoretical framework follows the psychological construct of subjective theories which are defined as systems of cognitions containing a rationale which is, at least, implicit (Groeben et al. 1988). The method depends on case studies. Data are collected by semi-structured interviews and interpreted according to the principles of classical hermeneutics. The construct of subjective theories and its adaption to the didactics of mathematics are briefly summed up by Eichler (2006).

In the following, a small part of this study will be presented. We will describe the difficulty of making sense of a teacher's utterances concerning geometrical applications. This difficulty was the initial point to *rethink the role of applications in elementary geometry in general*. Such a way of rethinking is one of the typical goals intended by the construct of subjective theories: This approach proposes, amongst others, to establish an exchange between individual opinions of "practising semi-specialists" and the theories of the scientific community.

A TEACHER'S OPINION ON APPLICATIONS IN GEOMETRY

The teacher of the case study presented here – let us call him Mr. B – has been taught mathematics, physical education, and computer science at a German secondary school for approximately 25 years. The age of his pupils ranges from 10 to 19 years. He seems to be well grounded in mathematics education and equipped with an elaborated concept of school-compatible mathematical applications. As a part of his position, he is involved in the education of trainee teachers in mathematics. This may be a further indication for the assumption that he is familiar with recent theories and perspectives of didactics.

As far as applied mathematics is concerned, his criteria for "good" applications match a lot of the attributes which are discussed and accepted by professional didacts (cf. Jablonka 1999). He demands that "the result [of a model building process] has to be useful for practical acting and reasoning" and that the real-world problems have to be "authentic and realistic, and not artificial and constructed" fulfilling their educational functions by being "challenging, but solvable – possibly after and due to simplification" (all quotations are translated by the author). He mentions the concepts of modelling and model building processes explicitly and approves the new style of arguing which is introduced to mathematics education by mathematization. He concludes: "Modelling and mathematical applications – these are things for which I would never abandon just a minute to discuss an automorphism instead."

AMIDST A STRUGGLE OF TENDENCIES?

At first sight, Mr. B seems to be a true advocate of model building processes and mathematization. But later, when asked how significant applications are for his everyday lessons taught in geometry, he admits that it is "not easy to find good geometrical applications." He refers to some examples taken from computer-aided design, navigation and traffic routing, but – as the main surprise – he does not expect that these applications are the ones his students should keep in mind. They should rather gain "an understanding of spatial relations" and forms and symmetries and they ought to deal with "rather simple applications" like drawing and folding figures or "reading a city map"; and finally, he does not ask which abilities can be conveyed by modelling and mathematization, but, instead, in which cases modelling is "more necessary for the students" – and one can add: to understand geometry.

At this point, there appears to be a rupture, possibly an inconsistency in Mr. B's perspectives concerning geometrical applications. On the one hand, he stresses the abilities and capacities in modelling and problem solving, which could be enforced by using authentic and challenging real-world problems; on the other hand, he regards "simple" geometrical applications as a tool to understand the concepts and theorems of elementary geometry – highlighting the knowledge of geometrical objects, of their attributes and dependencies as an educational goal on its own, and not as a device to manage practical challenges and to build up general skills beyond the scope of mathematics. The parts of goals and means seem to be suddenly switched over.

At first sight, there might be a simple and obvious explanation for Mr. B's ambivalent statements: He could be influenced by two different schools which Kaiser claims to have located within the discussion on mathematical applications (Kaiser 1995). She distinguishes between a *pragmatic* and a *scientific-humanistic approach*: In the pragmatic view, mathematics is a tool to solve practical problems. Applications are deemed as practices to achieve problem solving capacities in managing real-world issues (Kaiser 1995, p. 72). Therefore, applied mathematics is seen from a *procedural* point of view and modelling and model building *processes* are stressed as a content of the curriculum. The scientific-humanistic school, in contrast, emphasizes the principle of “conceptual mathematization”, that means that real-world situations are used to discover and develop mathematical concepts and insights and to receive mathematical ideas based on manifold associations (Kaiser 1995, p. 72).

GEOMETRICAL WORKING SPACES

To clarify the ideas of the scientific-humanistic school as far as geometry is concerned, it is suitable to use the theoretical framework of *geometrical working spaces* (summed up by Houdement 2007). By this approach, geometry is split into three different paradigms (Houdement & Kuzniak 2003):

1) Geometry I (Natural Geometry): Geometry is seen as an *empirical science* which refers to physical objects. To proof or to refute conjectures, both *deduction and experiments* are allowed, whereas measurement is the main experimental technique. This theory is *not axiomatic*, and its type of deduction is similar to inferential arguments between “local ordered” propositions in ordinary language discussions.

2) Geometry II (Natural Axiomatic Geometry): Geometry is treated as an *axiomatic theory*. The axioms are supposed to refer to the real world and, therefore, to describe physical figure and objects (with some idealization). Insofar, Geometry II is *empirical*, too. But to proof or to reject propositions, no empirical argument is permitted, but only a *deductive* one based on the axioms.

3) Geometry III (Formalist Axiomatic Geometry): Geometry is seen as an *axiomatic and deductive theory*, and no connection to the real world is intended.

With reference to this approach, the main goal of the scientific-humanistic school can be described as the project to prevent a sudden transition from Geometry I in primary school to Geometry III in the higher level secondary school in Germany. Such a sudden transition was enforced by the scientific tradition of this type of school and even increased by the New Maths movement until the early 1980s (Schupp 1994).

The alternative drift of the scientific-humanistic school was to fortify Geometry II, to establish a tender segue from Geometry I to II, and finally to achieve Geometry III or, at least, an idealistic interpretation of Geometry II which replaces the reference to physical objects by the platonic idea of *idealistic objects* not being present in the physical world. This project was mainly motivated by two reasons (cf. Kaiser 1995, p. 73): On the one hand, the ontological binding to real-world objects should be an *intermediate* stage on the way to an idealistic or formalist view of geometry to prevent a not understood formalism. On the other hand, it should establish an understanding of the role geometry plays as a tool in natural sciences. In both cases, the ontological foundation in real-world objects was primarily not intended to enforce model building processes and skills, but to build up a “field of associations” in order to understand geometry or natural science more proficiently.

NORMATIVE ISSUES OF APPLIED MATHEMATICS

Concerning applied mathematics, the pragmatic and scientific-humanistic approach differ in weighting *normative parameters*: One of them sets priorities in practical relevance und abilities to deal with model building processes; the other one stresses the theoretical aspects of mathematics (and natural sciences) and uses the associations to real-world situations as a tool to achieve a deep and connected understanding of mathematical concepts. The origin of this controversy appears to be nothing else but a disagreement about educational goals; and the different role of applications does not seem to arise from a specific character of geometry or geometrical applications, but only from disparate normative points of view – a situation which seems to have the same consequences in every part of mathematics and mathematics education, and not only in matters of geometry.

Exactly this opinion is called into question by our following considerations. We will propose an alternative assumption to explain the main statements of Mr. B. Our explanation is based on two arguments: Firstly, we will discuss an investigation on the *ontology* of geometry to clarify the question whether geometrical applications can be treated in the *same way* as other ones. Secondly, we will concern transcendental arguments to elaborate the issue to what extend the use and choice of geometrical applications are within the scope of *normative* deliberations.

THE STRUCTURAL THEORY OF EMPIRICAL SCIENCES

Our ontological consideration is influenced by a particular kind of *philosophy of science* which is called the "structuralist theory of empirical sciences", primarily established by Sneed and later elaborated by Stegmüller and others (Sneed 1979 and Stegmüller 1973/1985). The core assumption of this approach is the idea that empirical theories can be described by two components, namely by a *set-theoretical predicate* and a *set of intended applications* (Stegmüller 1973/1985, pp. 27–42). The set-theoretical predicate contains all of the formal and axiomatic aspects and is defined by the same method used by mathematicians in succession of Bourbaki: In the same manner, how it is possible to define the concept of a group as a pair (G,*) so that every element of G fulfils certain axioms relative to *, the axiomatic background of classical mechanics can be expressed by a quintuplet so that every element of the carrier set fulfils the well-known Newtonian axioms (Stegmüller 1973/1985, pp. 106–119).

At this stage, there is no difference between an empirical and a non-empirical theory (for example a mathematical theory from a formalistic point of view): They both can be defined by set-theoretical predicates. The difference arises from the set of intended applications: In case of non-empirical theories, this set is empty. In case of an empirical theory, it contains the applications which are claimed to be describable and explainable by the concerned theory. For instance, some of the intended applications of classical mechanics are pendulums, solar systems and especially apples falling from a tree. The set of intended applications cannot defined extensionally, but only by enumerating paradigmatic examples and by declaring that every entity also belongs to this set which is "sufficiently similar" to the paradigmatic examples – leaving vague what "sufficiently similar" means (Stegmüller 1973/1985, pp. 207–215).

The concept of geometrical working spaces is a useful framework to establish a connection between geometry and the structuralist theory of science: Geometry I and II are empirical theories insofar they are intended to refer to real-world objects, and they even share the same set of intended applications: physical objects of middle dimension, especially drawing figures and tinkered matters which are used at school. But despite sharing the same set of intended applications, these theories fundamentally differ in their set-theoretical predicates: Whereas Geometry II is assumed to fulfil an axiomatic system of Euclidean Geometry, the propositions of Geometry I may be so vague and psychologically motivated and so variable relative to different times and persons that they certainly cannot be transferred to a system of axioms and accordingly to a defining set-theoretical predicate. In contrast, Geometry III is not an empirical theory, since it is regarded in a formalist manner, presupposing not to have any applications; that means, in this case the set of intended application is empty. But on the other hand, Geometry III shares the same defining set-theoretical predicate with Geometry II: They both are intended to be a Euclidean Geometry.

The set of intended applications is not just an "illustration", a nice, but useless thing which can be left out; it rather fulfils two indispensable functions: From a logical point of view, the set of intended applications is a conceptual attribute and a part of the *definition of an empirical theory*. It distinguishes an empirical theory from a non-empirical one und declares the "part of the world" to which the theory is connected. Exactly this is the difference between Geometry II and III.

The second function results from the fact that every non-trivial empirical theory is based on idealization. For example, classical mechanics presupposes the existence of point particles without any spatial dimension. However, such entities do not exist in a strict sense of the word, but only "approximately" – and this is the second task of the set of intended applications: Since there is no way to explain explicitly under which condition and to what extent an approximation is allowed to make an empirical theory applicable (Stegmüller 1973/1985, pp. 207–215), i. e. under which condition an application belongs to the set of intended application, the paradigmatic examples of this set provides a number of "case studies" by which the *limits of approximation* are implicitly defined and novices of the scientific community can become familiar with the scope and borders of their coming occupation.

In geometry, the problem of approximation will typically arise, if infinity or dimension zero occurs; straight lines, planes, and angles are paradigmatic examples of this case (Struve 1990, p. 43). For instance, if there is a line drawn on a paper, there will be two ways to deal with the question "Is this a straight line, a segment of a straight line or neither of them?": From a formalist or idealistic view of geometry, this is a trivial question, since geometry does not refer to physical objects; a physical line is neither a segment nor straight line; at most, drawings could be symbolic tools to think about geometrical objects or propositions. But if it is taken serious that geometry can be interpreted as an empirical theory (as supposed in Geometry I and II and as being common and necessary for geometrical applications as we will see later), the pupils will have to learn to treat a line sometimes as a segment and sometimes as a straight line. To deal with these decisions is a notorious problem in geometry. The intended applications like drawing figures are the paradigmatic examples by which pupils are supposed to learn to manage these questions.

Hence, the knowledge of the set of indented applications and the handling of its vagueness is not optional, but an integral part of a particular empirical theory and, therefore, one of the aspects of "possessing" and being able to apply a certain theory. The educational task of paradigmatic examples is primarily described by Kuhn as far as philosophy of science is concerned (Kuhn 1962/1976, pp. 59–62). It is also a common thesis in psychology that paradigmatic examples play a major role in learning a theory (e. g. Seiler 2001, pp. 144–225).

ONTOLOGICAL ASPECTS OF ELEMENTARY GEOMETRY AT SCHOOL

At this point, we will come back to didactics. Struve has investigated how elementary geometry is presented in secondary school following the philosophy of science structuralism sketched above (Struve 1990, p. 6). Expressed in terms of the theory of geometrical working spaces, he comes to the conclusion that the didactical changes which were established to avoid a sudden switch from Geometry I to Geometry III by stressing Geometry II (as mentioned above) factually took the effect that the new textbooks present rather Geometry I than Geometry II and (even if Geometry II is reached) geometry is continuously taught as an empirical theory, and never as a formalistic or idealistic one as intended: "students learn an empirical theory in the geometry lessons held at secondary school" and "concerning the empirical theory, as we want to call the theory the students learn in their geometry lessons according to our investigation, figures created by folding and drawing are the paradigmatic examples" (Struve 1990, pp. 38–39).

THE ISSUE OF MODELLING

Struve has mentioned some of the consequences of his result – foremost some consideration on the fact that proofs have different functions in empirical and non-empirical theories observing that students typically treat proofs in the same manner as they are used in empirical sciences (Struve 1990, pp. 38–49). In this article, we will add a consideration concerning modelling. If we can follow Struve's results, Mr. B's distinction between two types of geometrical applications is not confusing, but an obvious implication of the empirical character of geometry as it is taught in secondary school: The figures created by drawing and folding and the "simple" applications based on these figures can be regarded as the paradigmatic examples which define the set of intended applications and constitute geometry as the empirical science of the spatial environment surrounding us in everyday life.

In this view, the supremacy of simple applications is not based on a normative decision about the role of application in mathematics education, but on the *specific ontology of geometry*: The knowledge of and the familiarity to these examples of applications are defining attributes of geometry as an empirical science. Hence, with regard to these "basic" applications, geometry differs from the other parts of mathematics taught at school. In the other cases, the amount and choice of applications is a normative question guided by arguments which Kaiser has combed through. In geometry, however, the task of normative deliberations begins not before the set of intended applications is left. Therefore, it is not astonishing that the (rare) cases which Mr. B mentions as "real" examples of modelling in geometry are quite different from the paradigmatic examples of folding and drawing: computer-aided design, navigation and traffic routing. In these cases and after some basic courses based on "simple" applications, geometry may no longer differ in modelling and mathematization.

TRANSCENDENTAL ASPECTS OF GEOMETRY

Our last task concerns the question if the dominance of an empirical view of geometry at school (as Geometry I or II) is an aberration caused by psychological circumstances and enforced by "misguided" textbooks or if there are good reasons to teach geometry as an empirical theory (to some extend). We will argue for the latter, accentuating a special role of geometry in contrast to other parts of mathematics and aiming for the conclusion that therefore two different types of applications are needed.

Let us start with an example: In 2003, a new national curriculum framework called "Bildungsstandards" (educational standards) was established in Germany. In contrast to former resolutions, this declaration stresses general skills, abilities and competencies – and among others, abilities in mathematical modelling. The relevant paragraph closes with the following sentence: "This includes translating the situation which is to be modelled into mathematical concepts, structures and relations" (KMK 2004, p. 8). This is a formulation ranging over all parts of mathematics taught at secondary school. A specific statement focussing on geometry is not declared.

Let us deliberate what this sentence presupposes: There is a real-world situation which can be described by mathematical concepts, but need not to be treated in this way. For instance, you can cross the road without thinking about the probability to be knocked over and you can look at the carps in a lake without having a function in mind to describe their growth process. Normally, a mathematical description is *not necessary* and will only be introduced, if it promises deeper insights as a description in ordinary language. Besides the general skills, this is a typical educational goal of modelling: the awareness that mathematics is a useful tool to achieve knowledge of the external world and to formulate this knowledge in a very precise manner.

In geometry, the case is quite different. If geometry could be treated like other mathematical theories, it would be possible to describe a situation geometrically *only on demand*. But this assumption fails since it is inevitable to use, at least, rudimental geometrical concepts to describe a situation at all. You cannot cross the road or look at the carps in the lake without possessing, at least, a broad understanding of basic geometrical concepts. For instance, a (vague) understanding of relative positions is necessary to individuate the different things, persons or objects which are part of a specific situation.

The idea that space is not a thing of human perception among others, but the conceptual framework which allows to describe real-world phenomena was primarily introduced by Kant as a part of his *transcendental* philosophy (Kant 1781/1998). In contemporary ontology the conceptual framework of space (and time) is broadly accepted as a condition to describe real-world situations (for everyday perceptions see Runggaldier and Kanzian 1998, pp. 17–52, as a condition of empirical sciences see Bartels 1996, pp. 23–71, or Stegmüller 1973/85, p. 60).

CONCLUSION: TWO TYPES OF GEOMETRICAL APPLICATIONS

Now, it is possible to connect both arguments: Following transcendental considerations, it is necessary to possess basic concepts to describe real-world situation and to establish the conditions under which model building processes are possible. That means, for mathematical reasons it may be passable to interpret geometry as a formalist or idealistic theory; but for model building processes or in contexts of natural sciences, it is necessary to understand geometry as an empirical theory. For some simple model building processes, an understanding on the level of Geometry I may be sufficient, but for more elaborated tasks or as a tool of natural sciences, Geometry II seems to be indispensable.

Against this background, we attain a “two step view” of geometrical applications: Since concepts of an empirical geometry are necessary to apply mathematics and, in a structuralist view of science, these concepts correspond to a set of intended applications taken from the world of folding and drawing, the first type of applications consists of very “simple” applications whose function is completely defined by learning and applying elementary geometry, especially by learning to manage the reference of concepts like “straight line” which can only be applied due to approximation. Hence, geometrical applications of a “simple” kind are *inevitable ingredients* of teaching geometry; and there is no reason to criticize the simplicity of these applications. At this stage, a normative debate about goals of teaching “applied geometry” is inadequate, since according to the empirical character of school geometry, there is no difference between teaching applied geometry and teaching geometry at all. This shall be our first conclusion: To some extend, it is necessary to deal with simple geometrical applications; and this necessity is not an inference from a normative decision about the goals of teaching applied mathematics, but a consequence of the specific ontological situation of geometry and it transcendental function as a condition of natural science and ordinary perception. No other part of secondary school mathematics possesses this ontological and transcendental function. For this reason, the status of geometry is unique, and the debate on geometrical applications cannot be held in the same way as it is possible in the scope of other parts of mathematics.

The second conclusion is related to the other type of geometrical applications: If the "simple" and intended applications are the only ones which students get to know, there will be an obvious deficit in teaching general skills and model building capacities in the sense of the pragmatic view of applied mathematics. Exactly this is the function of the second type of geometrical applications. It is comprehensible that applications which are intended to fulfil this task are quite different from the first ones. Mr. B mentions examples taken from computer-aided design, navigation and traffic routing. A list of similar examples is collected by Graumann (1994). Applications of this kind are typically not "pure geometrical", but includes concepts or hypotheses taken from natural or social sciences, basic economics or empirical tedium platitudes. This fact can be regarded as a further indication for our claim that there two different types of applications with distinct functions: Whereas the simple ones are used to built up geometrical concepts and to manage the vagueness of applying geometrical concepts to real-world situations, the more complex ones are intended to use *pre-existing* geometrical concepts and insights to reach some of the many educational goals which Kaiser sums up for model building processes in general (Kaiser 1995). For this purpose, a real-world problem only providing geometrical aspects often does not appear to be multifarious enough to allow a model building process whose challenges lie in this process (including mathematization, simplification, validation and hypothesis testing), and not in geometrical deliberations and calculations.

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