INTRODUCTION
GEOMETRICAL THINKING

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The Working Group 5 on Geometrical Thinking had around 30 participants from 14 countries all over Europe and from America too (Mexico, USA and Canada). During its sessions, the participants discussed 16 papers prepared for the Working Group and selected among 23 initial proposals and 15 have been retained for publication. The participants, and it's a strength of the group, worked within the continuity of the former sessions of Cerme. Some points can be considered as a common background known by ancient participants to the Working Group and the discussions among people were facilitated by this common culture. The readers are invited to have a look on the former general reports made at Bellaria (Dorier et al., 2003) and Larnaca (Kuzniak and al, 2007) when they want to know more about the common background.

This report insists on the questions of theoretical supports in Geometry, which can be seen as local theory in comparison of more general theoretical frameworks used in Mathematics Education. It would be interesting to explore the relationships between both local and global viewpoints. This part results from a collective work of a small group managed by Iliada Elia.

Then, all the accepted papers are briefly introduced for giving an idea of problems the group was concerned by.

Theoretical and methodological aspects of research in geometry

During the working group, we distinguished two approaches of using theory in research: First, theory can serve as a starting point for initiating a research study. For instance, the need to empirically validate or extend specific theories may motivate an investigation. Second, theory can act as a lens to look into the data. For example, different phenomena and behaviours observed in mathematics classes may evoke ideas to the teacher or the researcher for starting research. To start from phenomena or data is a valid first approach to research. In this case, theory may enable the teacher or the researcher to better understand and interpret the collected data.

Certainly, if one has a dual approach to research (data or theory) s/he can start with theory or data. This has methodological implications, that is, the methodology has to be appropriate to a chosen theory or to the collected data. The collection of data is very important, though, for both types of research. But to have substantial and long-standing effects to the research community's endeavour, the data, their use and interpretation should have a theoretical contribution (e.g. add or suggest modifications to an existing theory or develop new theory).

The most important theories in geometry education that were identified and discussed are the following: Van Hiele's levels, Geometrical Working Space and Geometrical paradigms and Duval's semiotic approach. Each line of theory approaches geometry learning from a different perspective and thus is helpful for different purposes. Van Hiele's theory is mainly helpful for evaluating students' reactions, productions and solutions to problems (phenomenological approach). Houdement and Kuzniak's (2003) theory about Geometrical Working Space and Geometrical Paradigms (e.g. Geometry I: Natural Geometry, Geometry II: Natural Axiomatic Geometry and Geometry III: Formal Axiomatic Geometry) is mainly helpful for classifying approaches, e.g. the types of argumentation used and to understand students' difficulties and errors (epistemological approach). Duval's (2005) theory is mainly helpful for examining the registers (e.g. geometrical figures, verbal representations-language) used in the field of geometry and their treatment in geometry tasks (semiotic approach).

Furthermore, there are psychological approaches to geometry that are often linked to spatial abilities, e.g. Gestalt and Piaget's theories, but are not very well taken into account in the mathematics education research community. Connecting these approaches with geometry theories and/or using them as a tool to look into the data in future studies could be a first step towards addressing this gap.

Future research on geometry theories and their articulation could use Geometrical Paradigms in a more operationalized manner to analyze existing curricula, to analyze students' behaviour and in investigating modelling and problem solving. Van Hiele's levels could be extended by proposing and empirically validating new (sub-)levels within their scale.

Educational goals and curriculum in geometry

The discussion on this general and fundamental topic was introduced by two papers. Using an epistemological approach, Boris Girnat criticized some present approaches in the learning of Geometry (especially in Germany) which leave aside the classical ontological aspect of Geometry. . He claims that there are two different types of applications in geometry and that they both are necessary and not exchangeable by each other: The first one contains simple applications which are paradigmatic examples to learn basic geometrical concepts; the second one includes more complex ones and refers to transcendental aspects.

Then Laurent Vivier and Alain Kuzniak described a French viewpoint on the Greek Geometrical Work at Secondary level. Beyond some similarities between France and Greece, it appears that the Euclidean tradition stays stronger in Greece but only for cultural reasons. Due to the lack of evaluation at the entrance on the university, the teaching of geometry is not viewed as important by the students and we can notice again the effects of evaluation on the real curriculum. In their study, the authors used a theoretical frame based on paradigms and geometrical working spaces and Greek people present in the group reacted and agreed with the conclusions. The presentation made at Cerme was thought as an important part of the research project.

Understanding and use of geometrical figures and diagrams

The study presented by Eleni Deliyianni investigated the role of various aspects of apprehension, i.e., perceptual, operative and discursive apprehension, in geometrical figure understanding. Based on a statistical exploration of data collected from 1086 primary and secondary school students, the existence of six main factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concepts. However, findings revealed differences between primary and secondary school students' performance and in the way they behaved during the solution of the tasks.

In her presentation Claudia Acuna used the old but always pertinent viewpoint on the treatment of geometric diagrams as Gestalt configurations. In geometry, the figural aspects of diagrams as symbols are used to solve problems. When figural information are treated, Gestalt configurations emerge: auxiliary figural configurations, real or virtual, that give meaning and substance to an idea that facilitates the proof or solution to the problem. In the paper, some arguments are given to acknowledge the existence of these resources.

Understanding and use of concepts and "proof" in geometry.

The work presented by Paola Vighi is concerned by the comparison of surfaces which need some mereological transformations in the sense of Duval. The same problems were given to two groups of pupils 10-11 years old having followed different ways of learning geometry: one traditional and the second more "experimental". She concludes with some observations about teaching geometry and suggestions for its improvement.

Caroline Bulf studied some symmetry's effects on conceptualization of new mathematical concept at two different levels at French secondary school, with students who are 12-13 years old and 14-15 y.o. From the study, the concept of symmetry makes students confused with the transformations of the plan introduced at the beginning of secondary school. Students seem to be more familiar with metrical properties relative to the symmetry and develop mathematical reasoning at the end of secondary school.

Mattheou Kallia investigated the basic geometrical knowledge of students of the Pedagogical Department of Education. She investigated mainly how they define similarity of shapes and how the intuitive knowledge affects their perception of similar shapes. The results showed that a large percentage of students are not in a position to correctly define the similarity of shapes and that initial intuition affects their responses and their mathematical achievement.

Two other papers were focused on the question of geometrical reasoning. Georgia Panoura and Athanasios Gagatsis underlined that the geometrical reasoning of primary and secondary school students can be compared mainly on the way students confronted and solved specific geometrical tasks: the strategies they used and the common errors appearing in their solutions. This comparison shed light to students' difficulties and phenomena related to the transition from Natural Geometry (the objects of this paradigm of geometry are material objects) to Natural Axiomatic Geometry (definitions and axioms are necessary to create the objects in this paradigm of geometry). They stressed the inconsistency of the didactical contract implied in primary and secondary school education and they conclude on the need for helping students progressively move from the geometry of observation to the geometry of deduction.

Based on a different framework, Taro Fujita seems to study the same problem in the case of geometry in Japan. This paper reports findings that indicate that as many as 80% of lower secondary age students can continue to consider that experimental verifications are enough to demonstrate that geometrical statements are true - even while, at the same time, understanding that proof is required to demonstrate that geometrical statements are true. Further data show that attending more closely to the matter of the 'Generality of proof' can disturb students' beliefs about experimental verification and make deductive proof meaningful for them. It could be interesting to interpret these results with the same tools as Panoura and Gagatsis: didactical contract and geometrical paradigms. It seems that the conclusions are very close but in different context.

Communication and assessment in geometry

In the two following papers, original tools were used to assess geometrical abilities and in the same time to help students in developing their skills in argumentation. Silvia Semana examined how the written report, within the context of assessment for learning, helps students in learning geometry and in developing their explanation and argumentation skills at the 8th grade in Portugal. This study suggests that using written reports improves those capabilities and, therefore, the comprehension of geometric concepts and processes. These benefits for learning are enhanced through the implementation of some assessment strategies, namely oral and written feedback.

Anat Levav developed an approach based on the presumption that solving mathematical problems in different ways may serve as a double role tool - didactical and diagnostic. She described a tool for the evaluation of the performance on multiple solution tasks (MST) in geometry. The tool is designed to enable the evaluation of subject's geometry knowledge and creativity as reflected from his solutions for a problem. The example provided for such evaluation is taken from an ongoing large-scale research aimed to examine the effectiveness of MSTs as a didactical tool. Anat Levav argued that this method could be extended to other domains in mathematics.

3D Geometry: Teaching, thinking and learning

The working group was concerned by some studies on 3D Geometry with new viewpoints due to the use of dynamical software in the learning of these specific parts of geometry which is often left aside in the real curriculum. Dynamic Geometry Environments (DGEs) in 2D are one of the well researched topics in mathematics education. DGEs for 3D-environments (Archimedes, Geo3D and Cabri 3D) were designed in Germany and France. Mathias Hattermann studied the specific drag-mode in 3D Geometry environments. He showed that pre-service teachers with previous knowledge in 2D-systems prefer to work with a real model of a cube instead of the 3D-system to solve certain problems. Previous knowledge in 2D-systems seems to be insufficient to handle the drag-mode in an appropriate way in 3D-environments. In a second study, he introduced the students to the special software before the investigation and distinguished different dragging modalities during the solution processes of two tasks.

The approach of Joris Mithalal is more on the transition to formal proof in 3D Geometry. Teaching mathematical proof is a great issue of mathematics education, and geometry is a traditional context for it. Nevertheless, especially in plane geometry, the students often focus on the drawings. As they can see results, they don't need to use neither axiomatic geometry nor formal proof. He tried to analyse how space geometry situations could incite students to use axiomatic geometry. Using Duval's distinctions between iconic and non-iconic visualization, he discussed the potentialities of situations based on a 3D dynamic geometry software.

In the two last papers, the authors focused on the traditional way of teaching and learning 3D Geometry. Edna Gonzalez presented part of the analysis of a Teaching Model for the geometry of solids of an initial Education Plan for elementary school teachers, and its implementation in the University School of Teaching of the Universitat de València in Spain.

In a statistical analysis of the results of 269 students (5^{th} to 9^{th} grade) in Cyprus, Marios Pittalis tried to show that 3D geometry thinking can be described across the following factors: (a) recognition and construction of nets, (b) representation of 3D objects, (c) structuring of 3D arrays of cubes, (d) recognition of 3D shapes' properties, (e) calculation of the volume and the area of solids, and (f) comparison of the properties of 3D shapes. With these factors, he identified four different profiles of students. In the future, it would be useful to make these kinds of studies in various contexts with other theoretical frameworks to validate the conclusions.

References

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