CAN YOU DO IT IN A DIFFERENT WAY?

Dina Tirosh, Pessia Tsamir & Michal Tabach

Tel Aviv University

In order to distinguish between two things one employs explicitly or implicitly a certain criterion. This criterion, being relevant to make the distinction in a given setting might be irrelevant in another setting. What counts as different in mathematics needs to be agreed upon. In this paper we analyze kindergarten children's different solutions to one task in order to learn about their ways of coping with multiple solutions and with multiple solution strategies. Our findings suggest that kindergarten children are able to suggest multiple solutions to this task and to apply several strategies to solve it, and that these abilities could be promoted by their engagement in related activities.

Let us start with a story about two kindergarten children, Nir and Jonathan, who were engaged in the Create an Equal Number (CEN) task. In this task, a child sat in a quiet corner of the kindergarten with an adult. He was presented with two distinct sets of bottle caps – three bottle caps were placed on one side of the table and five bottle caps were placed on the other (see Figure 1). All bottle caps had the same shape, size, and color. The child was asked: "Can you make it so that there will be an equal number of bottle caps on each side of the table?" After the child rearrange the bottle caps, the interviewer returned the bottle caps to their original arrangement (three in one set, five in the other) and asked the child, "Is there a different way to make the number of bottle caps on each side equal"? This rearrangement of the bottle cops (3 and 5) and the related question were repeated until the child said that there is no other way.

Figure 1: The initial stage of the CEN Task

The story of Nir: Nir looked closely at the two sets of bottle caps, and then he took out two caps from the set of five, and arranged each set of three in a similar position. In each set the caps were placed to formulate the vertices of an isosceles triangle. The interviewer then returned the caps to their original arrangement, asking Nir: "Is there a different way to make the number of bottle caps on each side equal"? Nir took out again two caps from the set of five, and this time he placed the caps in each set in a straight line, equally spread (see Figure 2).

Figure 2: Nir's second solution

Once more, the interviewer returned the setting to its original position, repeating his question. Again, Nir took out two caps from the set of five, rearranging the three caps in each set in a way similar to his first solution (isosceles triangles), but this time creating a larger distance between each pair of caps.

The interviewer rearranged the setting to its original position. Nir suggested a fourth solution, similar to his second solution (straight line), but this time with larger distances among the caps in each set (see Figure 3). In the following, and last iteration of the process, Nir provided the same solution as his first one.

Figure 3: Nir's forth solution

The story of Jonathan: Jonathan looked closely at the two sets of bottle caps, and then he took one cap from the set of five, and added it to the set of three. This act resulted in two sets with four bottles caps in each. Jonathan disregarded the actual arrangement of the caps in each set. The interviewer, then returned the caps to the original arrangement, asking Jonathan: "Is there a different way to make the number of bottle caps on each side equal"? Jonathan asked: "may I take caps out?" the interviewer approved, and Jonathan took out one cap from the set of three, and three caps from the set of five, creating two sets of two caps each.

Once more, the interviewer returned the setting to its original position, repeating his question. This time, Jonathan removed all the caps from both sets, saying "two sets of nothing".

The interviewer returned again the setting to its original position, and posed the question. Jonathan took out two caps from the set of five, creating two sets of three caps each. In the next iteration, Jonathan took out two caps from the set of three and four caps from the set of five, creating two sets of one cap each. In the last iteration Jonathan said: "there are no other options". It seemed that for Jonathan the spatial arrangement of the caps on the table was insignificant.

What can we learn from these two stories? The two children were engaged in the task and each of them provided several solutions, attempting to fulfill the interviewer's request for different solutions. Nir based his solutions on spatial attributes and differentiated between them in two ways: the relative placement of the caps in each set (a line shape versus a triangle shape), and the relative distance among the caps in each set. Note that in each of Nir's solutions there were three caps in each set, i.e., equal numbers of caps. Jonathan's solutions differed in one way: the (equal) number of bottle caps for each solution.

The solutions of the children were based on two main criteria: the spatial placement (figural arrangement, distance); the number of elements. Within mathematics discourse, each of these criteria can be considered as relevant for differentiating among solutions in a given context. A triangle may be considered different from a line when sorting geometrical figures. The distance among elements may be considered as a relevant criterion when comparing lengths. The number of elements is a criterion for differentiating quantities. Thus, the relevance of a given criterion as a means to differentiate among solutions is related to the task at hand and to the norms related to problem solving. These two issues are addressed in the theoretical background.

THEORETICAL BACKGROUND

During the last two decades there is a growing interest in early childhood mathematics education, and a growing recognition of its importance (e.g., NCTM, 2000; Sylva, Melhuish, Sammons, Siraj-Blatchford, & Taggett, 2004). NCTM recommends to provide children with activities aiming at promoting their mathematical thinking and understanding: "students understanding of mathematical ideas can be built throughout their school years if they actively engage in tasks and experiences designed to deepen and connect their knowledge" (NCTM, 2000, p. 21).

One way of promoting children's mathematical literacy is by engaging them in tasks with multiple solutions, and with a variety of related strategies: "opportunities to use strategies must be embedded naturally in the curriculum across the content areas" (NCTM, 2000, p. 54). The ability to identify differences and similarities among various strategies is context dependent and is by no means straight forward.

Yackel and Cobb (1996) highlighted the process of developing a common understanding of what counts as 'a different solution' in a classroom community. They claimed that "the sociomathematical norm of what constitutes mathematical difference supports higher-level cognitive activity" (p. 464). Establishing a socio-mathematical norm of what counts as different solution strategies is a key component in the creation of an autonomic learner.

Sfard and Levia (2005) analyzed a process in which Roni and Eynat, 4,0 and 4,7 year old, learned to interpret the term "the same" in a mathematical discourse with Roni's parents. Roni's mother presented the girls with two identical, closed boxes that contained marbles (the number of marbles could not be seen). She asked the girls "in which box are there more marbles? (p. 3)". To the mother's surprise, the girls chose one of the boxes, without attempting to count the number of marbles in the boxes. It was evident, from their reaction to the mother's later request to count, that both of them were capable of counting. When presented with two open boxes with the same number of marbles, upon the mother's request, the girls were able to count the marbles in each box, however did not use the term "the same" as an answer to the question "which box has more marbles?" Seven months later, the girls use counting as a strategy for comparing the number of marbles in the boxes on their own initiative, and they were also able to use the term "the same". Sfard and Levia concluded that the use of words in a mathematical setting needs to be learned by children.

In the present study, we examined 5-6 year old children's perceptions of "what counts as different and what counts as the same" in the context of the CEN task (creating two equivalent sets when presented with two unequivalent ones).

SETTING

Two groups of 5-6 year old children participated in this study. The first group consisted of 81 children, who were taught by teachers participating in a two-year, *Starting Right: Mathematics in Kindergarten* program (this program was initiated in Israel, in collaboration with the Rashi Foundation. Details about *Starting Right: Mathematics in Kindergarten* can be found in http://www.tafnit.org.il//pageframe.htm?page=http://www.tafnit.org.il/).
The CEN task and other such tasks were discussed with the Project-K-teachers. The project children worked on tasks from various mathematical domains, such as geometry, measurement, number and operations. Some of the tasks involved pictorial mediators, and others involved physical mediators, like the CEN task. We bring here as an illustration one other task.

The task dealt with the concept of equality, oriented to promote the children's understanding of equivalent sets. Four children sit in a quiet corner with their teacher. Each child had a set of cards and a game board. Some cards had printed items on, and the others had the equal sign on. The number of items on each card varied from one to ten. The drawings on each card consisted of identical items. Each quantity was represented on four different cards and there where different pictures on each card (Card 1: two cars, Card 2: two pencils, Card 3: two balls and card 4: two flowers). Each child in turn was expected to place the equal sign on the board, and then to choose from among his cards two cards which displayed an equal number of objects. The child was then expected to place the cards on the game board on both sides of the equal sign, creating a "mathematical sentence". The other children were expected to confirm or to reject the correctness of the "mathematical sentence", and explain their decisions. It was also possible to place more then one card on each side of the equal sign, as long as the total number of items on each side was equal.

The second group consisted of 82 children, who were taught by teachers who did not participate in the program.

All the children learned were from low socio-economic backgrounds in the same town. Jonathan was one of the project-group children, while Nir belonged to the other group.

The CEN Task analysis

In the CEN task, a child was individually presented in the initial stage with two sets of identical items. The sets differed in the number of elements. In other words, in the initial stage, children were presented with two unequivalent sets. Then, they were asked to create two sets with the same number of bottle caps. After a child offered a solution, the caps were rearranged in the original setting, and s/he was asked once more to create two sets with the same number of bottle caps. This process continued until the child responded that there are no more solutions. The way the situation was presented, and the wording of the request, implied that the critical criterion for "different and same" is the number of elements in each set.

Two characteristics of the task at hand may be somewhat unusual. First, the task has more than one solution. In fact, the task has five different solutions. Also, several strategies can be used to solve the task. Some are one step strategies: (a) *Taking from both* sets a number of elements, obtaining the same number of caps in each set. This strategy led to one of the following solutions: ((1;1) - i.e., one element in each set), (2;2). (b) *Removing all* the elements from both sets. This strategy led to the solution (0;0). (c) *Taking only from the larger* set, which, in our case, meant taking two elements from the set of five, obtaining the solution (3;3). (d) *Shifting from one set to the other,* which, in our case, led to the solution (4;4). A two-step strategy is (e) *Collecting all* the elements, and then creating two new sets "from scratch". The *collecting all* strategy could result in each of the five solutions of the task.

RESULTS AND DISCUSSION

First we report on the children's solutions, then on their solution strategies.

Solutions. As mentioned above, this task has five solutions. Table 1 shows that while 45% of the non project children came up with no more than one solution, 56% of the project children offered at least four solutions.

Table 1: The numbers of solutions per child (in %)

	No solution	One solution	Two solutions	Three solutions	Four solutions	Five solutions
Project (N=81)	2	6	15	21	37	19
Non-project (N=82)	7	38	12	16	20	7

Table 2 indicates that, the percentages of project children who suggested each solution was larger than those of the non-project children. The percentages in Table 2 may also point to the level of difficulty of each solution: the solution (4;4) was the easiest, (3;3) was somewhat harder, (2;2) and (1;1) were evidently harder. The cognitively problematic solution, consisting of empty sets (Linchevsky & Vinner, 1998), was employed only by 27% of the project children and 9% of the non-project children.

Table 2: The solutions provided by the children (in %)

	(0;0)	(1;1)	(2;2)	(3;3)	(4;4)
Project (N=81)	27	52	65	80	88
Non-project (N=82)	9	38	39	67	72

Solution strategies. While analyzing the task, we relate to five strategies that were used by the children, namely *take from both*, *remove all*, *taking only from the larger*, *shifting from one set to the other*, and *collecting all*. Table 3 presents the percentages of children from both groups who employed each strategy.

The strategy of shifting one cap from the set of five caps to the set of three caps was the dominant strategy for the children in both groups. Collecting all the elements from the two sets into one large set, and then creating two new, equal-number sets with some of the elements, was the least popular strategy.

Table 3: The strategies used by the children (in %)

	Shifting from one set to the other	Take only from the larger	Take from both	Remove all	Collect all
Project (N=81)	80	73	74	27	17
Non-project (N=82)	70	51	40	9	6

Table 3 also shows that each strategy was used by larger percentages of project children than non-project children. The *remove all* strategy was employed by 27% of the project children. This strategy requires special thinking, since the sets remained empty.

The percentages presented in Table 4 may suggest that most children used more than one strategy while working on the task. Table 4 presents the percentages of the number of different solution strategies used by the children.

Table 4: The number of solution strategies per child (in %)

	no strategy	One	Two	Three	Four	Five
Project (N=81)	2	9	25	44	19	1
Non-project (N=82)	7	44	23	17	9	--

About 90% of the project children employed more than one solution strategy while working on this task, and only about 50% of the non-project children did so. Children's ability to approach the task from several angels and to use more than one strategy is impressive.

SUMMING UP AND LOOKING AHEAD

The main focus of our study involved examining 5-6 year old children's perceptions of "what counts as different and what counts as the same" in the context the CEN task. This task has multiple solutions and multiple solution strategies. A task may include an unspoken constrain –all the caps should be used while creating the two sets. Maybe Jonathans' first solution was base on this constrain. When Jonathan was asked to find another solution, he explicitly asked "may I take caps out?" In this question, Jonathan might have expressed an understanding of the need to define the constrains of the task. Thus, he tried to find out the unspoken rules in this case. However, from Nir's behaviour we can learn that he did not have a similar constrains, and from his first solution he took out caps. Our data suggests that the project children outperformed their peers in the aspects we analyzed.

What could be concluded from the data presented here?

It seems that kindergarten children are capable of handling complex mathematical tasks, involving both multiple solutions and multiple solution strategies. The children provided creative solutions and employed creative solution-strategies. Silver (1997) argues that "mathematics educators can view creativity not as a domain of only a few exceptional individuals but rather as an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population" (p. 79). He relates to three core features of creativity in the context of problem solving: fluency, flexibility and novelty. Problems that are characterized by many solution methods, or answers, have the potential, according to Silver, to enhance two core components of students' creativity: fluency and flexibility.

Our data suggests that young students at the age of 5-6 year-old may already be engaged in such activities. Yet, many students who did not take part in the project, gave many solutions, and used a variety of solution strategies. At the same time, some project children did not displayed such behavior. This raises the questions: What determines a child's ability to provide several solutions? and What kind of experience may foster creative behavior?

In our study the two sets were presented with concrete materials (identical bottle caps). Gullen (1978) studied K-2nd students' strategies while comparing the number of elements in two sets, but he presented them pictorially. He found strong dependencies between the strategy used to compare the sets and students' grade levels, and also dependencies between the numbers of elements in the sets and the employed strategies. His findings suggest that students' performance may be depended on the task design.

More research is needed to identify parameters of tasks that may promote learning, i.e. presenting the task with concrete materials vs. presenting it pictorially? Starting from unequal, asking to create equal sets or starting with equal sets and asking to create unequal sets? Using homogenous elements or heterogeneous elements? Some other questions are: How many elements should be in each set? What other tasks can be presented to kindergartens to elicit several solution and several solutions strategies? What types of tasks could encourage children to identify the critical mathematical criteria that apply for a given setting?

REFERENCES

Gullen, G. E. (1978). Set comparison tactics and strategies of children in kindergarten, first grade, and second grade. *Journal for Research in Mathematics Education*, *9*(5), 349-360.

Linchevsky, L., & Vinner, S. (1998). The Naïve concept of sets in elementary teachers', *Proceedings of the 12th International Conference Psychology of Mathematics Education*, Vol. 1, pp. 471-478. Vezprem, Hungary.

National Council of Teachers of Mathematics [NCTM] (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Silver, E. A. (1997). Fostering creativity through instruction rich mathematical problem solving and problem posing. *ZDM The International Journal of Mathematics Education, 29*(3), 75-80.

Sfard, A., & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? - Early numerical thinking revisited. *Cognition and Instruction, 23*(2), 237-309.

Sylva, K., Melhuish, E., Sammons, P., Siraj-Blatchford, I., & Taggett, B. (2004). *The Effective provision of preschool education (EPPE) project: Final report. A longitudinal study funded by the DfES, 1997-2004*. London: DfES.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477.