EARLY YEARS MATHEMATICS – THE CASE OF FRACTIONS

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This paper describes children's understanding of order and equivalence of quantities represented by fractions, and their learning of fraction labels in part-whole and quotient situations. The study involves children aged 6 and 7 years who were not taught about fractions before. Two questions were addressed: (1) How do children understand the order and equivalence of quantities represented by fractions in quotient and part-whole situations? (2) Do children learn fraction labels more easily in one type of situation than another? Quantitative analysis showed that the situations in which the concept of fractions is used affected children's understanding of the quantities represented by fractions; their performance in quotient situations was better than in part-whole situations regarding order, equivalence and labelling.

This paper focuses on the effects of part-whole and quotient situations on children's understanding of the concept of fraction. It explores the impact of each of this type of situation on children's informal knowledge of fractions.

Framework

The Vergnaud's (1997) theory claims that to study and understand how mathematical concepts develop in children's minds through their experience in school and outside school, one must consider a concept as depending on three sets: a set of situations that make the concept useful and meaningful; a set of operational invariants used to deal with these situations; and a set of representations (symbolic, linguistic, graphical, etc.) used to represent invariants, situations and procedures. Following this theory, this paper describes a study on children's informal knowledge of quantities represented by fractions, focused on the effects of situations on children's understanding of the concept of fraction.

Literature distinguishes different classifications of situations that might offer a fruitful analysis of the concept of fractions. Kieren (1988, 1993) distinguished four types of situations – measure (which includes part-whole), quotient, ratio and operator - referred by the author as 'subconstructs' of rational number, considering a construct a collection of various elements of knowing; Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of rational number concept; Marshall (1993) distinguished five situations – part-whole, quotient, measures, operator, and ratio – based on the notion of 'schema' characterized as a network of knowledge about an event. More recently, Nunes, Bryant, Pretzlik, Evans, Wade and Bell (2004), based on the meaning of numbers in each situation, distinguished four situations – part-whole, quotient, operator and intensive quantities. In spite of the diversity, part-whole and quotient situations are distinguished in all these classifications. These situations were selected to be included in the study reported here.

In part-whole situations, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. So, 2/4 in a part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken. In quotient situations, the denominator designates the number of recipients and the numerator designates the number of items being shared. In a quotient situation, 2/4 means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situations a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction 2/4 can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each (Mack, 2001; Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004). Thus number meanings differ across situations. Therefore, it becomes relevant to know more about the effects of situations on children's understanding of fractions when building on their informal knowledge.

Applying Vergnaud's (1997) theory to the understanding of fractions, one also needs to consider a set of operational invariants that can be used in these situations. It is relevant to know under what condition children understand the relations between numerator, denominator and the quantity. The invariants analysed here are equivalence and ordering of the magnitude of fractions, more specifically, the inverse relation between the quotient and the magnitude.

Thus this study considers a set of situations (quotient, part-whole), a set of operational invariants (equivalence, ordering of fractional quantities), and a set of representations (symbolic, linguistic, pictorial) used to represent invariants, situations and procedures. This study investigates whether the situation in which the concept of fractions is used influences children's performance in problem solving tasks. The study was carried out with first-grade children who had not been taught about fractions in school. Two specific questions were investigated: (1) How do children understand the order and equivalence of fractions in part-whole and quotient situations? (2) Do children learn fraction labels differently in these situations?

Previous research (Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 2005) on children's understanding of division on sharing situations has shown that children aged 6 and 7 understand that, the larger the number of recipients, the smaller the part that each one receives, being able to order the values of the quotient. However, these studies were carried out with divisions in which the dividend was larger than the divisor. It is necessary to see whether the children will still understand the inverse relation between the divisor and the quotient when the result of the division would be a fraction. The study reported here tries to address these issues focusing on the qualitative understanding of this inverse relation. The equivalent insight using part-whole situations – the larger the number of parts into which a whole was cut, the smaller the size of the parts (Behr, Wachsmuth, Post & Lesh, 1984) – has not been documented in children of these age. Regarding equivalence in quotient situations, Empson (1999) found some evidence for children's use of ratios with concrete materials when children aged 6 and 7 years solved equivalence problems. In part-whole situations, Piaget, Inhelder and Szeminska (1960) found that children of this age level understand equivalence between the sum of all the parts and the whole and some of the slightly older children could understand the equivalence between parts, 1/2 and 2/4, if 2/4 was obtained by subdividing 1/2.

In a previous study, Mamede and Nunes (2008) compared the performance of 6 and 7 year-olds children when solving equivalence and ordering problems of quantities represented by fractions after being taught fraction labels in quotient, part-whole and operator situations. They found out that children who worked in quotient situations could succeed in some equivalence and ordering problems, but those who worked in part-whole and operator situations did not, despite all of them succeeded in labelling fractions. This shows that children are able to learn fraction labels without understanding the logic of fractions. The results of this study suggested that quotient situations were more suitable than the others when building on children's informal knowledge. Nevertheless, more research is needed regarding these issues.

Research about the impact of each of the situations in which fractions are used on the learning of fractions is difficult to find. Although some research has dealt with these situations with young children, these were not conceived to establish systematic and controlled comparisons between the situations. We still do not know much about the effects of each of these situations on children's understanding of fractions. Nevertheless, if we find out that there is a type of situation in which fractions make more sense for children, it would be a relevant finding to introduce fractions to them in the school. There have been no detailed comparisons between part-whole and quotient situations documented in research on children's understanding of fractions. This paper provides of such evidence.

METHOD

Participants

Portuguese first-grade children (N=80), aged 6 and 7 years, from the city of Braga, in Portugal, were assigned randomly to work in part-whole or quotient situations with the restriction that the same number of children in each level was assigned to each condition in each of the two schools involved in this study.

The children had not been taught about fractions in school, although the words 'metade' (half) and 'um-quarto' (a quarter) may have been familiar in other social settings.

The tasks

An example of a problem of equivalence and ordering presented to the children is given below on Tables 1 and 2.

Problems of equivalence of quantities represented by fractions	
Quotient situations	Part-whole situations
Two girls have to share 1 bar of chocolate fairly; 4 boys have to share 2 chocolates fairly. Does each girl eat the same, more, or less than each boy? Why do you think so?	Peter and Emma each have a bar of chocolate of the same size; Peter breaks his bar in 2 equal parts and eats 1 of them; Emma breaks hers into 4 equal parts and eats 2 of them. Does Peter eat more, the same, or less than Emma? Why do you think so?

Table 1: A problem of equivalence presented to the children in each type of situation.

Problems of ordering of quantities represented by fractions	
Quotient situations	Part-whole situations
Two boys have to share 1 bar of chocolate fairly; 3 girls have to share 1 chocolate bar fairly. Does each girl eat the same, more, or less than each boy? Why do you think so?	Bill and Ann each have a bar of chocolate of the same size; Bill breaks his bar into 2 equal parts and eats 1 of them; Ann breaks hers into 3 equal parts and eats 1 of them. Who eats more, Bill or Ann? Why do you think so?

Table 2: A problem of order presented to the children in each type of situation.

Regarding the labelling problems, there were two types: the ‘what fraction?’ problems, in which the child was asked to write the fractions that would represent the quantity; and the ‘inverse’ problem in which the fraction was given and the child was asked to identify the meaning of the numerator and denominator. An example of each type of labelling problems presented to the children is given below on Table 3.

Problem	Situation	Example
What fraction?	Part-whole	Paul is going to cut his chocolate bar into 4 equal parts and eats 3 of them. What fraction of the chocolate bar is Paul going to eat? Write the fraction in the box.
	Quotient	Three chocolate bars are going to be shared fairly among 4 friends. What fraction of chocolate does each

		friend eat? Write the fraction in the box.
Inverse	Part-whole	Anna divided her chocolate bar and ate 3/5 of it. Can you draw the chocolate bar and show how she did it?
	Quotient	Some children will share some chocolate bars. Each child gets 3/5 of the chocolate. How many children do you think there are? How many chocolates? Can you draw the children and the chocolates?

Table 3: An example of each type of labelling problems presented to the children in each type of situation.

Problems presented in part-whole situations were significantly longer than those presented in quotient situations. To reduce this effect, the interviewer made sure that each child understood the posed problem. All the problems were presented orally by the means of a story, with the support of computer slides. The children worked on booklets which contained drawings that illustrated the situations described. No concrete material was involved.

Design

At the beginning of the session, the six equivalence items and the six ordering items were presented in a block in random ordered. The children were seen individually by the experimenter. In the second part of the session, the children were taught how to label fractions with the unitary fractions 1/2, 1/3, 1/4 and 1/5 and the non-unitary fraction 2/3, in this order. After that, they were asked to solve three 'what fraction?' problems and one 'inverse' problem. All the numerical values were controlled for across situations.

RESULTS

Descriptive statistics for the performances on the tasks on quotient and part-whole situation are presented in Table 4.

	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Equivalence	2.1(1.5)	2.95 (1.54)	0.6 (0.7)	0.6 (0.5)
Ordering	3.3 (2.1)	4.25 (1.3)	1.45 (1.4)	1.2 (0.83)

Table 4: Mean (out of 6) and standard deviation (in brackets) of children's correct responses by task and situation.

A three-way mixed-model ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and problem solving situation (quotient *vs* part-whole) as between-participants factor, and tasks (Equivalence, Ordering) as within-participants factor.

There was a significant tasks effect, ($F(1,76)=18.54$, $p<.001$), indicating that children's performance on ordering tasks was better than in equivalence tasks. There was a significant main effect of the problem situation, ($F(1,76)=146.26$, $p< .001$), and a significant main effect of age, ($F(1,76)=4.84$, $p<.05$); there was a significant interaction of age by problem solving situation, ($F(1,76)=7.56$, $p<.05$). The older children performed better than the younger ones in quotient situations; in part-whole situations there was no age effect. There were no other significant effects.

An analysis of children's arguments was carried out and took into account all the productions, including drawings and verbalizations.

Based on the classifications of children's arguments when solving sharing problems (see Kornilaki & Nunes, 2005) and when solving equivalence problems in quotient situations (see Nunes et al., 2004), five types of arguments were distinguished attending to children's justifications solving equivalence and ordering problems in quotient situations, which were: a) invalid, comprising arguments that are not related to the problem; b) perceptual comparisons, the judgements are sustained on perceptual comparisons based on partitioning; c) valid argument, based on the inverse relation between the number of recipients and the size of the shares; d) only to the dividend (or numerator), based on the number of items to share and the shares, ignoring the inverse relation between the recipients and the shares; e) only to the divisor (or denominator), based on number of recipients and the shares, ignoring the number of items being shared.

Based on a classification of children's arguments on equivalence and ordering problems of fractions (see Behr et al., 1984), four arguments were distinguished also from children's justifications when solving equivalence and ordering problems, in part-whole situations. These four arguments were: a) invalid, comprising arguments that are not related to the problem; b) valid argument, based on the inverse relation between the number of parts into which the whole was cut and the number of parts eaten/taken, attending to the size of the shares; c) only to the dividend (or numerator), based on the number of parts eaten/taken, ignoring their sizes and the number of parts into which the whole was cut; d) only to the divisor (or denominator), based on the number of equal parts into which the whole was divide, ignoring their sizes and the number of parts eaten/taken.

Table 5 shows the children's arguments when solving equivalence and ordering problems and the rate of correct responses for problems in quotient and part-whole situations.

Children presented more valid arguments based on the inverse relation between the number of recipients and the size of the shares, when solving problems in quotient situations. In part-whole situations, the valid arguments were based on the inverse relation between the number of parts into which the whole was cut and the number of parts eaten/taken. In part-whole situations the most frequent arguments used were based on the number of parts eaten/taken, ignoring their sizes and the number of parts into which the whole was cut.

	Type of situation			
	Quotient (N=240)		Part-whole (N=240)	
Type of argument	Equiv.	Order	Equiv.	Order
Invalid	0	.01	.01	.02
Perceptual comparisons	.03	.09	-	-
Valid	.27	.38	.03	.06
Only to the dividend (numerator)	.09	.14	.18	.13
Only to the divisor (denominator)	.03	.01	.05	.01

Table 5: Type of argument and proportion of correct responses when solving the tasks in quotient and part-whole situations.

These results show that, when solving ordering problems in quotient situations, almost 40% of the responses were correct and justified with an explanation attending to the numerator, denominator and the quantity. This was not achieved when solving the correspondent problems in part-whole situations.

Also the fraction labels were analysed for each condition of study. Descriptive statistics for the performances on the labelling problems on quotient and part-whole situation are presented in Table 6.

	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Labelling	3.5(1.1)	3.5 (0.95)	2.3 (0.92)	2.4 (1.1)

Table 6: Mean (out of 4) and standard deviation (in brackets) of children's correct responses by task and situation.

In order to analyse the effect of situation on children's learning to label fractions, a two-factor ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and situation (quotient *vs* part-whole) as the main factors.

There was a significant main effect of situation, ($F(1,76)=25.45$, $p<.001$): children learned fractions labels more easily in quotient situations than in part-whole situations. There was no significant age effect and no interactions. Thus it can be concluded that the children learned to label fractions more easily in quotient situations than in part-whole situations and that is not dependent on age.

Figures 1 and 2 show examples of children's drawings when solving the inverse problems in quotient and part-whole situations, respectively. Some incorrect solutions will be shown and discussed in presentation.

Figure 1: Children's solution of the inverse problem in quotient situation.

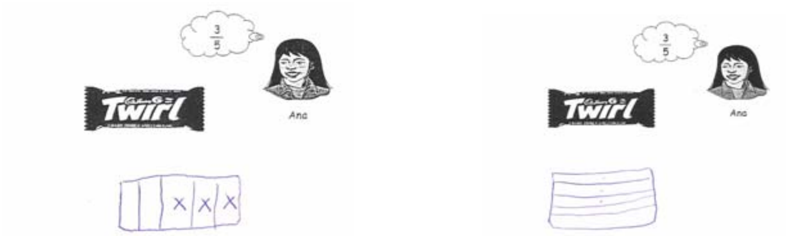

Figure 2: Children's solution of the inverse problem in part-whole situation.

These children were not taught about any strategies to solve the problems. In spite of succeeding in labelling problems in quotient and part-whole situations, only 30% of those who solved the inverse problem in part-whole situations drew the correct number of cuts and the correct number of parts taken. When dividing the chocolate bar, 37.5% of the children counted the number of cuts instead of the number of parts, ending up with the incorrect number of parts into which the whole was divided; 20% of the children drew incorrect number of cuts and incorrect number of parts taken, and 12.5% of the children could not to solve the problem. This contrasts with the 92.5% of children who successful solved the inverse problem in quotient situation, drawing the correct number of chocolates and the correct numbers of children; 2.5% drew the incorrect number of children but the correct number of chocolates, and 5% did not solve the problem.

DISCUSSION AND CONCLUSION

Children's ability to solve problems of equivalence and ordering of quantities represented by fractions is better in quotient than in part-whole situations. Children's arguments when solving these problems reveal that quotient situations are easier for the child to understand the relations between the numerator, denominator and the quantity. The levels of success on children's performance in quotient situations, supports the idea that children have some informal knowledge about equivalence and ordering of quantities represented by fractions. These results extend those obtained by Kornilaki and Nunes (2005), who showed that children aged 6 and 7 years succeeded on ordering problems, in sharing situations, where the dividend was larger than the divisor. The results presented here showed that the children still be able to use the same inverse reasoning when dealing with quantities represented by fractions. The findings of this study also extended those of Empson (1999) who showed that 6-7-year-olds children could solve equivalence and ordering problems in quotient situations, after being taught about equal sharing strategies. The children of this study were not taught about any strategies.

Regarding the labelling of fractions, the children's performance in both situations reveals that quotient situations are easier for children to master fraction labels, understanding the meaning of the numbers involved, than part-whole situations. In part-whole situations, the majority of the children also succeeded in labelling problems and understood the meaning of the numbers involved clearly enough to identify them in a new situation. These results converge with those found by Mamede and Nunes (2008) who showed that children of 6-7-year-olds could successful learn fractions labels in quotient and part-whole situations, understanding the meaning of the numbers involved, without being able to solve equivalence and ordering problems in these situations, having difficulties in understanding the relations between the numerator, denominator and the quantity.

In spite of succeeding in labelling fractions in both situations, the learning to label fractions in quotient and in part-whole situations seems to involve different types of difficulties for the children. Whereas in quotient situations the values involved in the fractions could easily be represented by drawing, as they refer to different variables – number of recipients and number of items being shared-, in part-whole situations, as both variables refer to parts, partitioning (division of a whole into equal parts) may play an important role for some children in this task.

This study shows that part-whole and quotient situations affect differently children's understanding of fractions. These results suggest that quotient situations should be explored in the classroom in the first years of school. Nevertheless, more research is needed providing a deeper insight on the effects of situations in which fractions are used on children's understanding of fractions.

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